1.1 A BRIEF PREVIEW OF CALCULUS: TANGENT LINES AND THE LENGTH OF A CURVE

In this section, we approach the boundary between precalculus mathematics and the calculus by investigating several important problems requiring the use of calculus. Recall that the slope of a straight line is the change in y divided by the change in x. This fraction is the same regardless of which two points you use to compute the slope. For example, the points $(0, 1)$, $(1, 4)$, and $(3, 10)$ all lie on the line $y = 3x + 1$. The slope of 3 can be obtained from any two of the points. For instance,

$$m = \frac{4-1}{1-0} = 3 \quad \text{or} \quad m = \frac{10-1}{3-0} = 3.$$

In the calculus, we generalize this problem to find the slope of a *curve* at a point. For instance, suppose we wanted to find the slope of the curve $y = x^2 + 1$ at the point $(1, 2)$. You might think of picking a second point on the parabola, say $(2, 5)$. The slope of the line through these two points (called a **secant line**; see Figure 1.2a) is easy enough to compute. We have

$$m_{sec} = \frac{5-2}{2-1} = 3.$$

However, using the points $(0, 1)$ and $(1, 2)$, we get a different slope (see Figure 1.2b):

$$m_{sec} = \frac{2-1}{1-0} = 1.$$

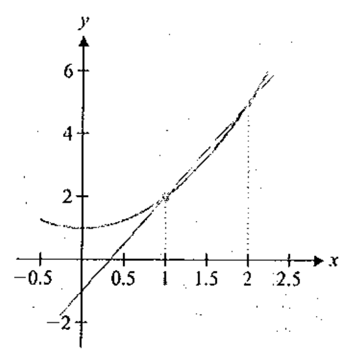

FIGURE 1.2a
Secant line, slope = 3

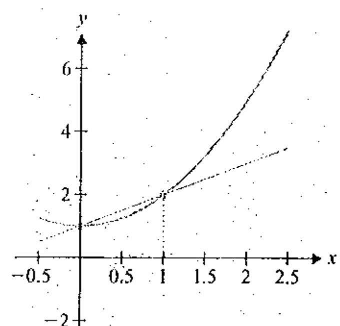

FIGURE 1.2b
Secant line, slope = 1

For curves other than straight lines, the slopes of secant lines joining different points are generally *not* the same, as seen in Figures 1.2a and 1.2b.

If you get different slopes using different pairs of points, then what exactly does it mean for a curve to have a slope at a point? The answer can be visualized by graphically zooming in on the specified point. Take the graph of $y = x^2 + 1$ and zoom in tight on the point $(1, 2)$. You should get a graph something like the one in Figure 1.3. The graph looks very much like a straight line. In fact, the more you zoom in, the straighter the curve appears to be and the less it matters which two points are used to compute a slope. So, here's the strategy: pick several points on the parabola, each closer to the point $(1, 2)$ than the previous one. Compute the slopes of the lines through $(1, 2)$ and each of the points. The closer the second point gets to $(1, 2)$, the closer the computed slope is to the answer you seek.

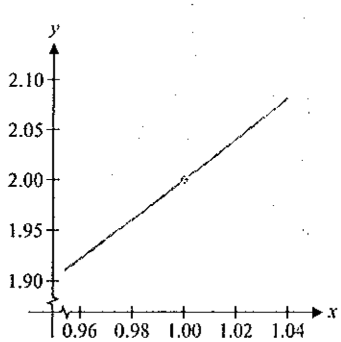

FIGURE 1.3
$y = x^2 + 1$

For example, the point (1.5, 3.25) is on the parabola fairly close to (1, 2). The slope of the line joining these points is

$$m_{sec} = \frac{3.25 - 2}{1.5 - 1} = 2.5.$$

The point (1.1, 2.21) is even closer to (1, 2). The slope of the secant line joining these two points is

$$m_{sec} = \frac{2.21 - 2}{1.1 - 1} = 2.1.$$

Continuing in this way, observe that the point (1.01, 2.0201) is closer yet to the point (1, 2). The slope of the secant lines through these points is

$$m_{sec} = \frac{2.0201 - 2}{1.01 - 1} = 2.01.$$

The slopes of the secant lines (2.5, 2.1, 2.01) are getting closer and closer to the slope of the parabola at the point (1, 2). Based on these calculations, it seems reasonable to say that the slope of the curve is approximately 2.

Example 1.1 takes our introductory example just a bit further.

EXAMPLE 1.1 Estimating the Slope of a Curve

Estimate the slope of $y = x^2 + 1$ at $x = 1$.

Solution We focus on the point whose coordinates are $x = 1$ and $y = 1^2 + 1 = 2$. To estimate the slope, choose a sequence of points near (1, 2) and compute the slopes of the secant lines joining those points with (1, 2). (We showed sample secant lines in Figures 1.2a and 1.2b.) Choosing points with $x > 1$ (x-values of 2, 1.1 and 1.01) and points with $x < 1$ (x-values of 0, 0.9 and 0.99), we compute the corresponding y-values using $y = x^2 + 1$ and get the slopes shown in the following table.

Second Point	m_{sec}
(2, 5)	$\frac{5-2}{2-1} = 3$
(1.1, 2.21)	$\frac{2.21-2}{1.1-1} = 2.1$
(1.01, 2.0201)	$\frac{2.0201-2}{1.01-1} = 2.01$

Second Point	m_{sec}
(0, 1)	$\frac{1-2}{0-1} = 1$
(0.9, 1.81)	$\frac{1.81-2}{0.9-1} = 1.9$
(0.99, 1.9801)	$\frac{1.9801-2}{0.99-1} = 1.99$

Observe that in both columns, as the second point gets closer to (1, 2), the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point (1, 2) is then 2. ■

In Chapter 2, we develop a powerful technique for computing such slopes exactly (and easily). Note what distinguishes the calculus problem from the corresponding algebra problem. The calculus problem involves a process we call a *limit*. While we presently can only estimate the slope of a curve using a sequence of approximations, the limit allows us to compute the slope exactly.

EXAMPLE 1.2 Estimating the Slope of a Curve

Estimate the slope of $y = \sin x$ at $x = 0$.

Solution This turns out to be a very important problem, one that we will return to later. For now, choose a sequence of points near (0, 0) and compute the slopes of the secant lines joining those points with (0, 0). The following table shows one set of choices.

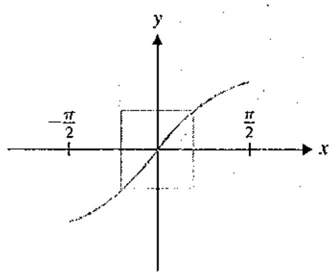

FIGURE 1.4
$y = \sin x$

Second Point	m_{sec}
(1, sin 1)	0.84147
(0.1, sin 0.1)	0.99833
(0.01, sin 0.01)	0.99998

Second Point	m_{sec}
(−1, sin (−1))	0.84147
(−0.1, sin (−0.1))	0.99833
(−0.01, sin (−0.01))	0.99998

Note that as the second point gets closer and closer to (0, 0), the slope of the secant line (m_{sec}) appears to get closer and closer to 1. A good estimate of the slope of the curve at the point (0, 0) would then appear to be 1. Although we presently have no way of computing the slope exactly, this is consistent with the graph of $y = \sin x$ in Figure 1.4. Note that near (0, 0), the graph resembles that of $y = x$, a straight line of slope 1. ■

A second problem requiring the power of calculus is that of computing distance along a curved path. While this problem is of less significance than our first example (both historically and in the development of the calculus), it provides a good indication of the need for mathematics beyond simple algebra. You should pay special attention to the similarities between the development of this problem and our earlier work with slope.

Recall that the (straight-line) distance between two points (x_1, y_1) and (x_2, y_2) is

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For instance, the distance between the points (0, 1) and (3, 4) is

$$d\{(0, 1), (3, 4)\} = \sqrt{(3 - 0)^2 + (4 - 1)^2} = 3\sqrt{2} \approx 4.24264.$$

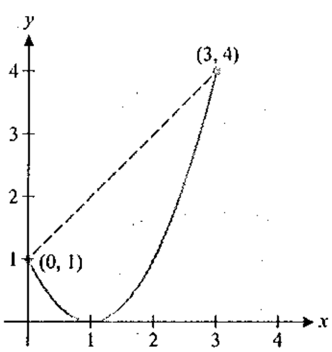

FIGURE 1.5a
$y = (x - 1)^2$

However, this is not the only way we might want to compute the distance between these two points. For example, suppose that you needed to drive a car from (0, 1) to (3, 4) along a road that follows the curve $y = (x - 1)^2$ (see Figure 1.5a). In this case, you don't care about the straight-line distance connecting the two points, but only about how far you must drive along the curve (the *length* of the curve or *arc length*).

Notice that the distance along the curve must be greater than $3\sqrt{2}$ (the straight-line distance). Taking a cue from the slope problem, we can formulate a strategy for obtaining a sequence of increasingly accurate approximations. Instead of using just one line segment to get the approximation of $3\sqrt{2}$, we could use two line segments, as in Figure 1.5b. Notice that the sum of the lengths of the two line segments appears to be a much better approximation to the actual length of the curve than the straight-line distance used previously. This

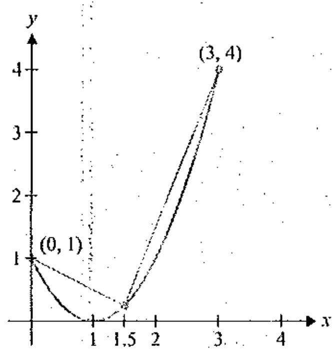

FIGURE 1.5b
Two line segments

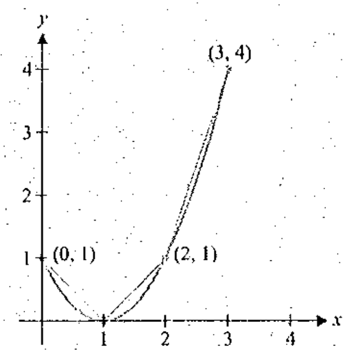

FIGURE 1.5c
Three line segments

distance is

$$d_2 = d\{(0,\ 1), (1.5, 0.25)\} + d\{(1.5, 0.25), (3, 4)\}$$
$$= \sqrt{(1.5 - 0)^2 + (0.25 - 1)^2} + \sqrt{(3 - 1.5)^2 + (4 - 0.25)^2} \approx 5.71592.$$

You're probably way ahead of us by now. If approximating the length of the curve with two line segments gives an improved approximation, why not use three or four or more? Using the three line segments indicated in Figure 1.5c, we get the further improved approximation

$$d_3 = d\{(0, 1), (1, 0)\} + d\{(1, 0), (2, 1)\} + d\{(2, 1), (3, 4)\}$$
$$= \sqrt{(1 - 0)^2 + (0 - 1)^2} + \sqrt{(2 - 1)^2 + (1 - 0)^2} + \sqrt{(3 - 2)^2 + (4 - 1)^2}$$
$$= 2\sqrt{2} + \sqrt{10} \approx 5.99070.$$

No. of Segments	*Distance*
1	4.24264
2	5.71592
3	5.99070
4	6.03562
5	6.06906
6	6.08713
7	6.09711

Note that the more line segments we use, the better the approximation appears to be. This process will become much less tedious with the development of the definite integral in Chapter 4. For now we list a number of these successively better approximations (produced using points on the curve with evenly spaced x-coordinates) in the table found in the margin. The table suggests that the length of the curve is approximately 6.1 (quite far from the straight-line distance of 4.2). If we continued this process using more and more line segments, the sum of their lengths would approach the actual length of the curve (about 6.126). As in the problem of computing the slope of a curve, the exact arc length is obtained as a limit.

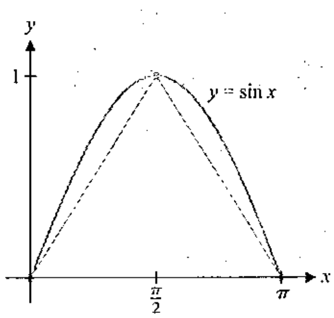

FIGURE 1.6a
Approximating the curve with two line segments

EXAMPLE 1.3 Estimating the Arc Length of a Curve

Estimate the arc length of the curve $y = \sin x$ for $0 \le x \le \pi$ (see Figure 1.6a).

Solution The endpoints of the curve on this interval are $(0, 0)$ and $(\pi, 0)$. The distance between these points is $d_1 = \pi$. The point on the graph of $y = \sin x$ corresponding to the midpoint of the interval $[0, \pi]$ is $(\pi/2, 1)$. The distance from $(0, 0)$ to $(\pi/2, 1)$ plus the distance from $(\pi/2, 1)$ to $(\pi, 0)$ (illustrated in Figure 1.6a) is

$$d_2 = \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} + \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \approx 3.7242.$$

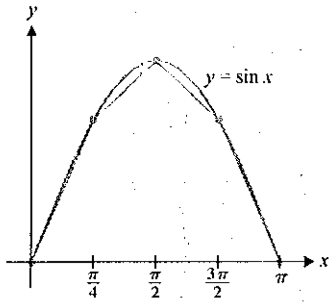

FIGURE 1.6b
Approximating the curve with four line segments

Number of Line Segments	*Sum of Lengths*
8	3.8125
16	3.8183
32	3.8197
64	3.8201

Using the five points $(0, 0)$, $(\pi/4, 1/\sqrt{2})$, $(\pi/2, 1)$, $(3\pi/4, 1/\sqrt{2})$, and $(\pi, 0)$ (i.e., four line segments, as indicated in Figure 1.6b), the sum of the lengths of these line segments is

$$d_4 = 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}} + 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} \approx 3.7901.$$

Using nine points (i.e., eight line segments), you need a good calculator and some patience to compute the distance of 3.8125. A table showing further approximations is given in the margin. At this stage, it would be reasonable to estimate the length of the sine curve on the interval $[0, \pi]$ as slightly more than 3.8.

BEYOND FORMULAS

In the process of estimating both the slope of a curve and the length of a curve, we make some reasonably obvious (straight-line) approximations and then systematically improve on those approximations. In each case, the shorter the line segments are, the closer the approximations are to the desired value. The essence of this is the concept of *limit*, which separates precalculus mathematics from the calculus. At first glance, this limit idea might seem of little practical importance, since in our examples we never compute the exact solution. In the chapters to come, we will find remarkably simple shortcuts to exact answers. Can you think of ways to find the exact slope in example 1.1?

EXERCISES 1.1

WRITING EXERCISES

1. Explain why each approximation of arc length in example 1.3 is less than the actual arc length.

2. To estimate the slope of $f(x) = x^2 + 1$ at $x = 1$, you would compute the slopes of various secant lines. Note that $y = x^2 + 1$ curves up. Explain why the secant line connecting (1, 2) and (1.1, 2.21) will have slope greater than the slope of the tangent line. Discuss how the slope of the secant line between (1, 2) and (0.9, 1.81) compares to the slope of the tangent line.

In exercises 1–12, estimate the slope (as in example 1.1) of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 + 1, a = 1$
2. $f(x) = x^2 + 1, a = 2$
3. $f(x) = \cos x, a = 0$
4. $f(x) = \cos x, a = \pi/2$
5. $f(x) = x^3 + 2, a = 1$
6. $f(x) = x^3 + 2, a = 2$
7. $f(x) = \sqrt{x+1}, a = 0$
8. $f(x) = \sqrt{x+1}, a = 3$
9. $f(x) = e^x, a = 0$
10. $f(x) = e^x, a = 1$
11. $f(x) = \ln x, a = 1$
12. $f(x) = \ln x, a = 2$

In exercises 13–20, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

13. $f(x) = x^2 + 1, 0 \le x \le 2$
14. $f(x) = x^3 + 2, 0 \le x \le 1$
15. $f(x) = \cos x, 0 \le x \le \pi/2$
16. $f(x) = \sin x, 0 \le x \le \pi/2$
17. $f(x) = \sqrt{x+1}, 0 \le x \le 3$
18. $f(x) = 1/x, 1 \le x \le 2$
19. $f(x) = x^2 + 1, -2 \le x \le 2$
20. $f(x) = x^3 + 2, -1 \le x \le 1$

21. An important problem in calculus is finding the area of a region. Sketch the parabola $y = 1 - x^2$ and shade in the region above the x-axis between $x = -1$ and $x = 1$. Then sketch in the following rectangles: (1) height $f(-\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = -1$ to $x = -\frac{1}{2}$. (2) height $f(-\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = -\frac{1}{2}$ to $x = 0$. (3) height $f(\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = 0$ to $x = \frac{1}{2}$. (4) height $f(\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = \frac{1}{2}$ to $x = 1$. Compute the sum of the areas of the rectangles. Based on your sketch, does this give you a good approximation of the area under the parabola?

22. To improve the approximation of exercise 21, divide the interval $[-1, 1]$ into 8 pieces and construct a rectangle of the appropriate height on each subinterval. Compared to the approximation in exercise 21, explain why you would expect this to be a better approximation of the actual area under the parabola.

23. Use a computer or calculator to compute an approximation of the area in exercise 21 using (a) 16 rectangles, (b) 32 rectangles, (c) 64 rectangles. Use these calculations to conjecture the exact value of the area under the parabola.

24. Use the technique of exercises 21–23 to estimate the area below $y = \sin x$ and above the x-axis between $x = 0$ and $x = \pi$.

25. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 1$.

26. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 2$.

EXPLORATORY EXERCISE

1. Several central concepts of calculus have been introduced in this section. An important aspect of our future development of calculus is to derive simple techniques for computing quantities such as slope and arc length. In this exercise, you will learn how to directly compute the slope of a curve at a point. Suppose you want the slope of $y = x^2$ at $x = 1$. You could start by computing slopes of secant lines connecting the point $(1, 1)$ with nearby points. Suppose the nearby point has x-coordinate $1 + h$, where h is a small (positive or negative) number. Explain why the corresponding y-coordinate is $(1 + h)^2$. Show that the slope of the secant line is $\frac{(1+h)^2 - 1}{1 + h - 1} = 2 + h$. As h gets closer and closer to 0, this slope better approximates the slope of the tangent line. Letting h approach 0, show that the slope of the tangent line equals 2. In a similar way, show that the slope of $y = x^2$ at $x = 2$ is 4 and find the slope of $y = x^2$ at $x = 3$. Based on your answers, conjecture a formula for the slope of $y = x^2$ at $x = a$, for any unspecified value of a.

1.2 THE CONCEPT OF LIMIT

In this section, we develop the notion of limit using some common language and illustrate the idea with some simple examples. The notion turns out to be a rather subtle one, easy to think of intuitively, but a bit harder to pin down in precise terms. We present the precise definition of limit in section 1.6. There, we carefully define limits in considerable detail. The more informal notion of limit that we introduce and work with here and in sections 1.3, 1.4 and 1.5 is adequate for most purposes.

As a start, consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

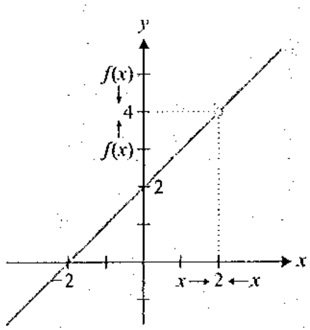

FIGURE 1.7a

$y = \frac{x^2 - 4}{x - 2}$

Notice that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x? We often find important clues about the behavior of a function from a graph (see Figures 1.7a and 1.7b).

Notice that the graphs of these two functions look quite different in the vicinity of $x = 2$. Although we can't say anything about the value of these functions at $x = 2$ (since this is outside the domain of both functions), we can examine their behavior in the vicinity of

21. An important problem in calculus is finding the area of a region. Sketch the parabola $y = 1 - x^2$ and shade in the region above the x-axis between $x = -1$ and $x = 1$. Then sketch in the following rectangles: (1) height $f(-\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = -1$ to $x = -\frac{1}{2}$, (2) height $f(-\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = -\frac{1}{2}$ to $x = 0$, (3) height $f(\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = 0$ to $x = \frac{1}{2}$, (4) height $f(\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = \frac{1}{2}$ to $x = 1$. Compute the sum of the areas of the rectangles. Based on your sketch, does this give you a good approximation of the area under the parabola?

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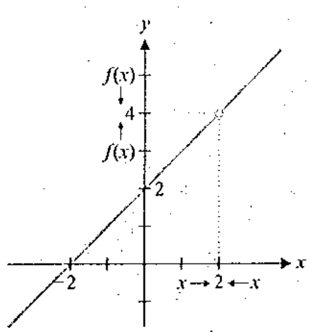

FIGURE 1.7a
$y = \frac{x^2 - 4}{x - 2}$

Notice that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x? We often find important clues about the behavior of a function from a graph (see Figures 1.7a and 1.7b).

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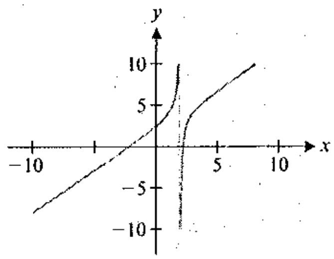


FIGURE 1.7b
$$y = \frac{x^2 - 5}{x - 2}$$

this point. We consider these functions one at a time. First, for $f(x) = \dfrac{x^2 - 4}{x - 2}$, we compute some values of the function for x close to 2, as in the following tables.

x	$f(x) = \dfrac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x) = \dfrac{x^2 - 4}{x - 2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

Notice that as you move down the first column of the table, the x-values get closer to 2, but are all less than 2. We use the notation $x \to 2^-$ to indicate that *x approaches 2 from the left side.* Notice that the table and the graph both suggest that as x gets closer and closer to 2 (with $x < 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the left** is 4, written

$$\lim_{x \to 2^-} f(x) = 4.$$

Likewise, we need to consider what happens to the function values for x close to 2 but larger than 2. Here, we use the notation $x \to 2^+$ to indicate that *x approaches 2 from the right side.* We compute some of these values in the second table.

Again, the table and graph both suggest that as x gets closer and closer to 2 (with $x > 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the right** is 4, written

$$\lim_{x \to 2^+} f(x) = 4.$$

We call $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ **one-sided limits.** Since the two one-sided limits of $f(x)$ are the same, we summarize our results by saying that the **limit of $f(x)$ as x approaches 2** is 4, written

$$\lim_{x \to 2} f(x) = 4.$$

The notion of limit as we have described it here is intended to communicate the behavior of a function *near* some point of interest, but not actually *at* that point. We finally observe that we can also determine this limit algebraically, as follows. Notice that since the expression in the numerator of $f(x) = \dfrac{x^2 - 4}{x - 2}$ factors, we can write

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} && \text{Cancel the factors of } (x - 2). \\ &= \lim_{x \to 2}(x + 2) = 4, && \text{As } x \text{ approaches 2, } (x + 2) \text{ approaches 4.} \end{aligned}$$

where we can cancel the factors of $(x - 2)$ since in the limit as $x \to 2$, x is *close* to 2, but $x \neq 2$, so that $x - 2 \neq 0$.

x	$g(x) = \frac{x^2-5}{x-2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10,003.9999

x	$g(x) = \frac{x^2-5}{x-2}$
2.1	−5.9
2.01	−95.99
2.001	−995.999
2.0001	−9995.9999

Similarly, we consider one-sided limits for $g(x) = \dfrac{x^2-5}{x-2}$, as $x \to 2$. Based on the graph in Figure 1.7b and the table of approximate function values shown in the margin, observe that as x gets closer and closer to 2 (with $x < 2$), $g(x)$ increases without bound. Since there is no number that $g(x)$ is approaching, we say that the *limit of $g(x)$ as x approaches 2 from the left does not exist,* written

$$\lim_{x\to 2^-} g(x) \textit{ does not exist.}$$

Similarly, the graph and the table of function values for $x > 2$ (shown in the margin) suggest that $g(x)$ decreases without bound as x approaches 2 from the right. Since there is no number that $g(x)$ is approaching, we say that

$$\lim_{x\to 2^+} g(x) \textit{ does not exist.}$$

Finally, since there is no common value for the one-sided limits of $g(x)$ (in fact, neither limit exists), we say that the *limit of $g(x)$ as x approaches 2 does not exist,* written

$$\lim_{x\to 2} g(x) \textit{ does not exist.}$$

Before moving on, we should summarize what we have said about limits.

A limit exists if and only if both corresponding one-sided limits exist and are equal. That is,

$$\lim_{x\to a} f(x) = L, \text{ for some number } L, \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

In other words, we say that $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close as we might like to L, by making x sufficiently close to a (on either side of a), but not equal to a.

Note that we can think about limits from a purely graphical viewpoint, as in example 2.1.

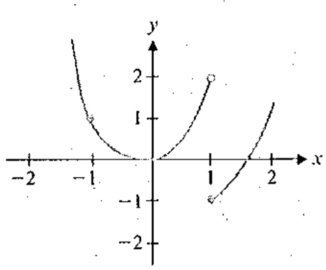


FIGURE 1.8
$y = f(x)$

EXAMPLE 2.1 Determining Limits Graphically

Use the graph in Figure 1.8 to determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$.

Solution For $\lim_{x\to 1^-} f(x)$, we consider the y-values as x gets closer to 1, with $x < 1$. That is, we follow the graph toward $x = 1$ *from the left* ($x < 1$). Observe that the graph dead-ends into the open circle at the point (1, 2). Therefore, we say that $\lim_{x\to 1^-} f(x) = 2$. For $\lim_{x\to 1^+} f(x)$, we follow the graph toward $x = 1$ *from the right* ($x > 1$). In this case, the graph dead-ends into the solid circle located at the point (1, −1). For this reason, we say that $\lim_{x\to 1^+} f(x) = -1$. Because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, we say that $\lim_{x\to 1} f(x)$ does not exist. Finally, we have that $\lim_{x\to -1} f(x) = 1$, since the graph approaches a y-value of 1 as x approaches −1 both from the left and from the right. ■

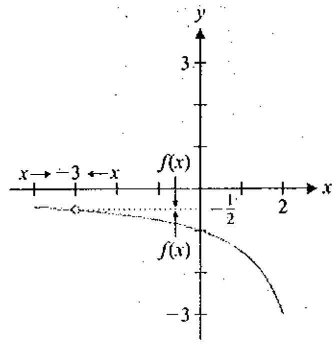


FIGURE 1.9
$\lim_{x\to-3}\frac{3x+9}{x^2-9}=-\frac{1}{2}$

x	$\frac{3x+9}{x^2-9}$
−3.1	−0.491803
−3.01	−0.499168
−3.001	−0.499917
−3.0001	−0.499992

x	$\frac{3x+9}{x^2-9}$
−2.9	−0.508475
−2.99	−0.500835
−2.999	−0.500083
−2.9999	−0.500008

EXAMPLE 2.2 A Limit Where Two Factors Cancel

Evaluate $\lim_{x\to-3}\frac{3x+9}{x^2-9}$.

Solution We examine a graph (see Figure 1.9) and compute some function values for x near −3. Based on this numerical and graphical evidence, it's reasonable to conjecture that

$$\lim_{x\to-3^+}\frac{3x+9}{x^2-9}=\lim_{x\to-3^-}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$

Further, note that

$$\lim_{x\to-3^-}\frac{3x+9}{x^2-9}=\lim_{x\to-3^-}\frac{3(x+3)}{(x+3)(x-3)} \quad \text{Cancel factors of } (x+3)$$
$$=\lim_{x\to-3^-}\frac{3}{x-3}=-\frac{1}{2},$$

since $(x-3)\to-6$ as $x\to-3$. Again, the cancellation of the factors of $(x+3)$ is valid since in the limit as $x\to-3$, x is *close* to −3, but $x\neq-3$, so that $x+3\neq0$. Likewise,

$$\lim_{x\to-3^+}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$

Finally, since the function approaches the *same* value as $x\to-3$ both from the right and from the left (i.e., the one-sided limits are equal), we write

$$\lim_{x\to-3}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$ ■

In example 2.2, the limit exists because both one-sided limits exist and are equal. In example 2.3, neither one-sided limit exists.

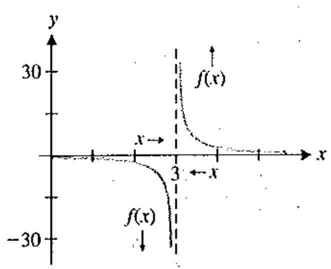


FIGURE 1.10
$y=\frac{3x+9}{x^2-9}$

x	$\frac{3x+9}{x^2-9}$
3.1	30
3.01	300
3.001	3000
3.0001	30,000

EXAMPLE 2.3 A Limit That Does Not Exist

Determine whether $\lim_{x\to3}\frac{3x+9}{x^2-9}$ exists.

Solution We first draw a graph (see Figure 1.10) and compute some function values for x close to 3.

Based on this numerical and graphical evidence, it appears that, as $x\to3^+$, $\frac{3x+9}{x^2-9}$ is increasing without bound. Thus,

$$\lim_{x\to3^+}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Similarly, from the graph and the table of values for $x<3$, we can say that

$$\lim_{x\to3^-}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Since neither one-sided limit exists, we say

$$\lim_{x\to3}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Here, we considered both one-sided limits for the sake of completeness. Of course, you should keep in mind that if *either* one-sided limit fails to exist, then the limit does not exist. ■

x	$\frac{3x+9}{x^2-9}$
2.9	−30
2.99	−300
2.999	−3000
2.9999	−30,000

Many limits cannot be resolved using algebraic methods. In these cases, we can approximate the limit using graphical and numerical evidence, as we see in example 2.4.

EXAMPLE 2.4 Approximating the Value of a Limit

Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution Unlike some of the limits considered previously, there is no algebra that will simplify this expression. However, we can still draw a graph (see Figure 1.11) and compute some function values.

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.99999983
0.0001	0.9999999983
0.00001	0.999999999983

x	$\frac{\sin x}{x}$
−0.1	0.998334
−0.01	0.999983
−0.001	0.99999983
−0.0001	0.9999999983
−0.00001	0.999999999983

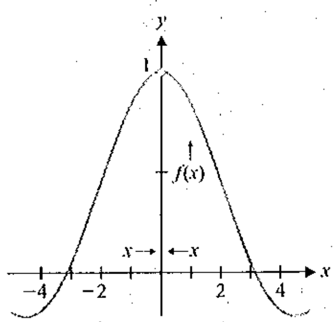


FIGURE 1.11
$\lim_{x\to 0} \frac{\sin x}{x} = 1$

The graph and the tables of values lead us to the conjectures:

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$

from which we conjecture that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In Chapter 2, we examine these limits with greater care (and prove that these conjectures are correct). ■

REMARK 2.1

Computer or calculator computation of limits is unreliable. We use graphs and tables of values only as (strong) evidence pointing to what a plausible answer might be. To be certain, we need to obtain careful verification of our conjectures. We see how to do this in sections 1.3–1.7.

EXAMPLE 2.5 A Case Where One-Sided Limits Disagree

Evaluate $\lim_{x\to 0} \frac{x}{|x|}$.

Solution The computer-generated graph shown in Figure 1.12a is incomplete. Since $\frac{x}{|x|}$ is undefined at $x = 0$, there is no point at $x = 0$. The graph in Figure 1.12b correctly shows open circles at the intersections of the two halves of the graph with the y-axis. We also have

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} \quad \text{Since } |x| = x, \text{ when } x > 0.$$
$$= \lim_{x\to 0^+} 1$$
$$= 1$$

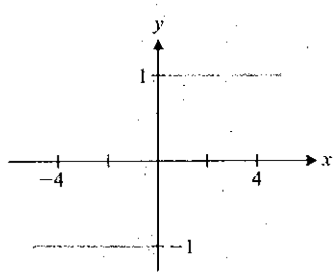


FIGURE 1.12a
$y = \frac{x}{|x|}$

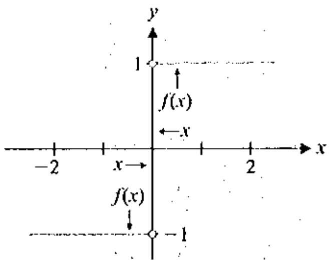


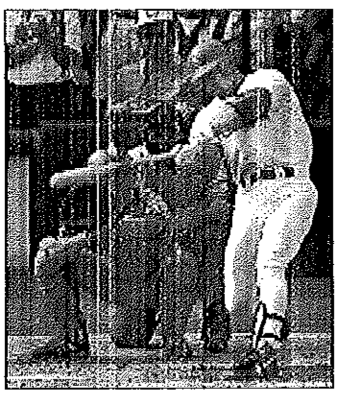
FIGURE 1.12b
$\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

and
$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0^-} -1 = -1.$$

It now follows that
$$\lim_{x\to 0} \frac{x}{|x|} \text{ does not exist,}$$

since the one-sided limits are not the same. You should also keep in mind that this observation is entirely consistent with what we see in the graph. ■

EXAMPLE 2.6 A Limit Describing the Movement of a Baseball Pitch



The knuckleball is one of the most exotic pitches in baseball. Batters describe the ball as unpredictably moving left, right, up and down. For a typical knuckleball speed of 60 mph, the left/right position of the ball (in feet) as it crosses the plate is given by

$$f(\omega) = \frac{1.7}{\omega} - \frac{5}{8\omega^2}\sin(2.72\omega)$$

(derived from experimental data in Watts and Bahill's book *Keeping Your Eye on the Ball*), where ω is the rotational speed of the ball in radians per second and where $f(\omega) = 0$ corresponds to the middle of home plate. Folk wisdom among baseball pitchers has it that the less spin on the ball, the better the pitch. To investigate this theory, we consider the limit of $f(\omega)$ as $\omega \to 0^+$. As always, we look at a graph (see Figure 1.13) and generate a table of function values. The graphical and numerical evidence suggests that $\lim_{\omega\to 0^+} f(\omega) = 0$.

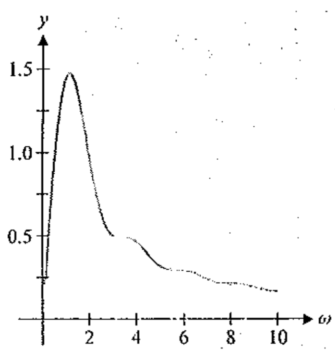


FIGURE 1.13
$y = \frac{1.7}{\omega} - \frac{5}{8\omega^2}\sin(2.72\omega)$

ω	$f(\omega)$
10	0.1645
1	1.4442
0.1	0.2088
0.01	0.021
0.001	0.0021
0.0001	0.0002

The limit indicates that a knuckleball with absolutely no spin doesn't move at all (and therefore would be easy to hit). According to Watts and Bahill, a very slow rotation rate of about 1 to 3 radians per second produces the best pitch (i.e., the most movement). Take another look at Figure 1.13 to convince yourself that this makes sense. ■

EXERCISES 1.2

WRITING EXERCISES

1. Suppose your professor says, "You can think of the limit of $f(x)$ as x approaches a as *what $f(a)$ should be*." Critique this statement. What does it mean? Does it provide important insight? Is there anything misleading about it? Replace the phrase in italics with your own best description of what the limit is.

2. Your friend's professor says, "The limit is a *prediction* of what $f(a)$ will be." Compare and contrast this statement to the one in exercise 1. Does the inclusion of the word *prediction* make the limit idea seem more useful and important?

3. We have observed that $\lim_{x\to a} f(x)$ does not depend on the actual value of $f(a)$, or even on whether $f(a)$ exists. In principle, functions such as $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 13 & \text{if } x = 2 \end{cases}$ are as "normal" as functions such as $g(x) = x^2$. With this in mind, explain why it is important that the limit concept is independent of how (or whether) $f(a)$ is defined.

4. The most common limit encountered in everyday life is the *speed limit*. Describe how this type of limit is very different from the limits discussed in this section.

---

1. For the function graphed below, identify each limit or state that it does not exist.

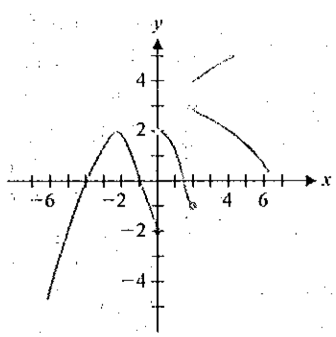
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 1^-} f(x)$

(e) $\lim_{x\to -1} f(x)$ (f) $\lim_{x\to 2^-} f(x)$

(g) $\lim_{x\to 2^+} f(x)$ (h) $\lim_{x\to 2} f(x)$

(i) $\lim_{x\to -2} f(x)$ (j) $\lim_{x\to 3} f(x)$



2. For the function graphed below, identify each limit or state that it does not exist.

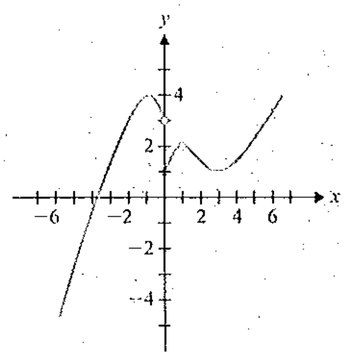
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x\to -2} f(x)$ (f) $\lim_{x\to 1^-} f(x)$

(g) $\lim_{x\to 1^+} f(x)$ (h) $\lim_{x\to 1} f(x)$

(i) $\lim_{x\to -1} f(x)$ (j) $\lim_{x\to 3} f(x)$



3. Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$ (d) $\lim_{x\to 1} f(x)$

4. Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x+1} - 2 & \text{if } x > 0 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to -1} f(x)$ (e) $\lim_{x\to 3} f(x)$

5. Sketch the graph of $f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$ and identify each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 1} f(x)$

6. Sketch the graph of $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 \le x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$ and identify each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$ (c) $\lim_{x\to -1} f(x)$
(d) $\lim_{x\to 1} f(x)$ (e) $\lim_{x\to 0} f(x)$

7. Evaluate $f(1.5)$, $f(1.1)$, $f(1.01)$ and $f(1.001)$, and conjecture a value for $\lim_{x\to 1^+} f(x)$ for $f(x) = \frac{x-1}{\sqrt{x}-1}$. Evaluate $f(0.5)$, $f(0.9)$, $f(0.99)$ and $f(0.999)$, and conjecture a value for $\lim_{x\to 1^-} f(x)$ for $f(x) = \frac{x-1}{\sqrt{x}-1}$. Does $\lim_{x\to 1} f(x)$ exist?

8. Evaluate $f(-1.5)$, $f(-1.1)$, $f(-1.01)$ and $f(-1.001)$, and conjecture a value for $\lim_{x\to -1^-} f(x)$ for $f(x) = \frac{x+1}{x^2-1}$. Evaluate $f(-0.5)$, $f(-0.9)$, $f(-0.99)$ and $f(-0.999)$, and conjecture a value for $\lim_{x\to -1^+} f(x)$ for $f(x) = \frac{x+1}{x^2-1}$. Does $\lim_{x\to -1} f(x)$ exist?

In exercises 9–14, use numerical and graphical evidence to conjecture values for each limit.

9. $\lim_{x\to 1} \frac{x^2-1}{x-1}$
10. $\lim_{x\to -1} \frac{x^2+x}{x^2-x-2}$
11. $\lim_{x\to 0} \frac{x^2+x}{\sin x}$
12. $\lim_{x\to \pi} \frac{\sin x}{x-\pi}$
13. $\lim_{x\to 0} \frac{\tan x}{\sin x}$
14. $\lim_{x\to 0} e^{-1/x^2}$

In exercises 15–26, use numerical and graphical evidence to conjecture whether the limit at $x = a$ exists. If not, describe what is happening at $x = a$ graphically.

15. $\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1}$
16. $\lim_{x\to -1} \frac{x^2-1}{x+1}$
17. $\lim_{x\to 1} \frac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$
18. $\lim_{x\to 0} \frac{x^2+4x}{\sqrt{x^3+x^2}}$
19. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$
20. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$
21. $\lim_{x\to 2} \frac{x-2}{|x-2|}$
22. $\lim_{x\to -1} \frac{|x+1|}{x^2-1}$
23. $\lim_{x\to 0} \ln x$
24. $\lim_{x\to 0} x\ln(x^2)$
25. $\lim_{x\to 0} \frac{\tan x}{x}$
26. $\lim_{x\to 0} \frac{\tan^{-1} x}{x}$

27. Compute $\lim_{x\to 1} \frac{x^2+1}{x-1}$, $\lim_{x\to 2} \frac{x+1}{x^2-4}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are polynomials with $g(a) = 0$ and $f(a) \ne 0$. What can you conjecture about $\lim_{x\to a} \frac{f(x)}{g(x)}$?

28. Compute $\lim_{x\to -1} \frac{x+1}{x^2+1}$, $\lim_{x\to \pi} \frac{\sin x}{x}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are functions with $f(a) = 0$ and $g(a) \ne 0$. What can you conjecture about $\lim_{x\to a} \frac{f(x)}{g(x)}$?

In exercises 29–32, sketch a graph of a function with the given properties.

29. $f(-1) = 2$, $f(0) = -1$, $f(1) = 3$ and $\lim_{x\to 1} f(x)$ does not exist.

30. $f(x) = 1$ for $-2 \le x \le 1$, $\lim_{x\to 1^+} f(x) = 3$ and $\lim_{x\to -2} f(x) = 1$.

31. $f(0) = 1$, $\lim_{x\to 0^-} f(x) = 2$ and $\lim_{x\to 0^+} f(x) = 3$.

32. $\lim_{x\to 0} f(x) = -2$, $f(0) = 1$, $f(2) = 3$ and $\lim_{x\to 2} f(x)$ does not exist.

33. As we see in Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is given by $m = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$. Estimate the slope m. Graph $y = \sqrt{x}$ and the line with slope m through the point $(1, 1)$.

34. As we see in Chapter 2, the velocity of an object that has traveled $\sqrt{x}$ miles in x hours at the $x = 1$ hour mark is given by $v = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$. Estimate this limit.

35. Consider the following arguments concerning $\lim_{x\to 0^+} \sin\frac{\pi}{x}$. First, as $x > 0$ approaches 0, $\frac{\pi}{x}$ increases without bound; since $\sin t$ oscillates for increasing t, the limit does not exist. Second: taking $x = 1, 0.1, 0.01$ and so on, we compute $\sin\pi = \sin 10\pi = \sin 100\pi = \cdots = 0$; therefore the limit equals 0. Which argument sounds better to you? Explain. Explore the limit and determine which answer is correct.

36. Consider the following argument concerning $\lim_{x\to 0} e^{-1/x}$. As x approaches 0, $\frac{1}{x}$ increases without bound and $\frac{-1}{x}$ decreases without bound. Since e^t approaches 0 as t decreases without bound, the limit equals 0. Discuss all the errors made in this argument.

37. Numerically estimate $\lim_{x\to 0^+}(1+x)^{1/x}$ and $\lim_{x\to 0^-}(1+x)^{1/x}$. Note that the function values for $x > 0$ increase as x decreases, while for $x < 0$ the function values decrease as x increases. Explain why this indicates that, if $\lim_{x\to 0}(1+x)^{1/x}$ exists, it is between function values for positive and negative x's. Approximate this limit correct to eight digits.

38. Explain what is wrong with the following logic (you may use exercise 37 to convince yourself that the answer is wrong, but discuss the logic without referring to exercise 37): as $x \to 0$, it is clear that $(1+x) \to 1$. Since 1 raised to any power is 1, $\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}(1)^{1/x} = 1$.

39. Numerically estimate $\lim_{x\to 0^+} x^{\sec x}$. Try to numerically estimate $\lim_{x\to 0^-} x^{\sec x}$. If your computer has difficulty evaluating the function for negative x's, explain why.

40. Explain what is wrong with the following logic (note from exercise 39 that the answer is accidentally correct): since 0 to any power is 0, $\lim_{x\to 0} x^{\sec x} = \lim_{x\to 0} 0^{\sec x} = 0$.

41. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function g such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

42. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists and $f(0)$ exists, but $\lim_{x\to 0} f(x) \neq f(0)$.

43. In the text, we described $\lim_{x\to a} f(x) = L$ as meaning "as x gets closer and closer to a, $f(x)$ is getting closer and closer to L." As x gets closer and closer to 0, it is true that x^2 gets closer and closer to -0.01, but it is certainly not true that $\lim_{x\to 0} x^2 = -0.01$. Try to modify the description of limit to make it clear that $\lim_{x\to 0} x^2 \neq -0.01$. We explore a very precise definition of limit in section 1.6.

44. In Figure 1.13, the final position of the knuckleball at time $t = 0.68$ is shown as a function of the rotation rate ω. The batter must decide at time $t = 0.4$ whether to swing at the pitch. At $t = 0.4$, the left/right position of the ball is given by $h(\omega) = \frac{1}{\omega} - \frac{5}{8\omega^2}\sin(1.6\omega)$. Graph $h(\omega)$ and compare to Figure 1.13. Conjecture the limit of $h(\omega)$ as $\omega \to 0$. For $\omega = 0$, is there any difference in ball position between what the batter sees at $t = 0.4$ and what he tries to hit at $t = 0.68$?

45. A parking lot charges \$2 for each hour or portion of an hour, with a maximum charge of \$12 for all day. If $f(t)$ equals the total parking bill for t hours, sketch a graph of $y = f(t)$ for $0 \le t \le 24$. Determine the limits $\lim_{t\to 3.5} f(t)$ and $\lim_{t\to 4} f(t)$, if they exist.

46. For the parking lot in exercise 45, determine all values of a with $0 \le a \le 24$ such that $\lim_{t\to a} f(t)$ does not exist. Briefly discuss the effect this has on your parking strategy (e.g., are there times where you would be in a hurry to move your car or times where it doesn't matter whether you move your car?).

EXPLORATORY EXERCISES

1. In a situation similar to that of example 2.6, the left/right position of a knuckleball pitch in baseball can be modeled by $P = \frac{5}{8\omega^2}(1 - \cos 4\omega t)$, where t is time measured in seconds ($0 \le t \le 0.68$) and ω is the rotation rate of the ball measured in radians per second. In example 2.6, we chose a specific t-value and evaluated the limit as $\omega \to 0$. While this gives us some information about which rotation rates produce hard-to-hit pitches, a clearer picture emerges if we look at P over its entire domain. Set $\omega = 10$ and graph the resulting function $\frac{1}{160}(1 - \cos 40t)$ for $0 \le t \le 0.68$. Imagine looking at a pitcher from above and try to visualize a baseball starting at the pitcher's hand at $t = 0$ and finally reaching the batter, at $t = 0.68$. Repeat this with $\omega = 5$, $\omega = 1$, $\omega = 0.1$ and whatever values of ω you think would be interesting. Which values of ω produce hard-to-hit pitches?

2. In this exercise, the results you get will depend on the accuracy of your computer or calculator. Work this exercise and compare your results with your classmates' results. We will investigate $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$. Start with the calculations presented in the table (your results may vary):

x	$f(x)$
0.1	−0.499583...
0.01	−0.49999583...
0.001	−0.4999999583...

Describe as precisely as possible the pattern shown here. What would you predict for $f(0.0001)$? $f(0.00001)$? Does your computer or calculator give you this answer? If you continue trying powers of 0.1 (0.000001, 0.0000001 etc.) you should eventually be given a displayed result of −0.5. Do you think this is exactly correct or has the answer just been rounded off? Why is rounding off inescapable? It turns out that −0.5 is the exact value for the limit, so the round-off here is somewhat helpful. However, if you keep evaluating the function at smaller and smaller values of x, you will eventually see a reported function value of 0. This round-off error is not so benign; we discuss this error in section 1.7. For now, evaluate $\cos x$ at the current value of x and try to explain where the 0 came from.

1.3 COMPUTATION OF LIMITS

Now that you have an idea of what a limit is, we need to develop some means of calculating limits of simple functions. In this section, we present some basic rules for dealing with common limit problems. We begin with two simple limits.

39. Numerically estimate $\lim_{x\to 0^+} x^{\sec x}$. Try to numerically estimate $\lim_{x\to 0^-} x^{\sec x}$. If your computer has difficulty evaluating the function for negative x's, explain why.

40. Explain what is wrong with the following logic (note from exercise 39 that the answer is accidentally correct): since 0 to any power is 0, $\lim_{x\to 0} x^{\sec x} = \lim_{x\to 0} 0^{\sec x} = 0$.

41. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function g such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

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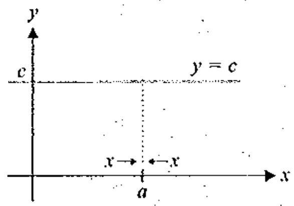

FIGURE 1.14
$\lim_{x \to a} c = c$

For any constant c and any real number a,

$$\lim_{x \to a} c = c. \tag{3.1}$$

In other words, the limit of a constant is that constant. This certainly comes as no surprise, since the function $f(x) = c$ does not depend on x and so, stays the same as $x \to a$ (see Figure 1.14). Another simple limit is the following.

For any real number a,

$$\lim_{x \to a} x = a. \tag{3.2}$$

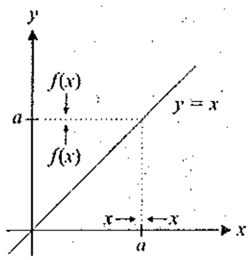

FIGURE 1.15
$\lim_{x \to a} x = a$

Again, this is not a surprise, since as $x \to a$, x will approach a (see Figure 1.15). Be sure that you are comfortable enough with the limit notation to recognize how obvious the limits in (3.1) and (3.2) are. As simple as they are, we use them repeatedly in finding more complex limits. We also need the basic rules contained in Theorem 3.1.

THEOREM 3.1

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and let c be any constant. The following then apply:

(i) $\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$,

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$ and

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ $\left(\text{if } \lim_{x \to a} g(x) \neq 0\right)$.

The proof of Theorem 3.1 is found in Appendix A and requires the formal definition of limit discussed in section 1.6. You should think of these rules as sensible results that you would certainly expect to be true, given your intuitive understanding of what a limit is. Read them in plain English. For instance, part (ii) says that the limit of a sum (or a difference) equals the sum (or difference) of the limits, *provided the limits exist.* Think of this as follows. If as x approaches a, $f(x)$ approaches L and $g(x)$ approaches M, then $f(x) + g(x)$ should approach $L + M$.

Observe that by applying part (iii) of Theorem 3.1 with $g(x) = f(x)$, we get that, whenever $\lim_{x \to a} f(x)$ exists,

$$\lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x) \cdot f(x)]$$
$$= \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} f(x)\right] = \left[\lim_{x \to a} f(x)\right]^2.$$

Likewise, for any positive integer n, we can apply part (iii) of Theorem 3.1 repeatedly, to yield

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \tag{3.3}$$

(see exercises 67 and 68).

Notice that taking $f(x) = x$ in (3.3) gives us that for any integer $n > 0$ and any real number a,

$$\lim_{x \to a} x^n = a^n. \tag{3.4}$$

That is, to compute the limit of any positive power of x, you simply substitute in the value of x being approached.

EXAMPLE 3.1 Finding the Limit of a Polynomial

Apply the rules of limits to evaluate $\lim_{x \to 2}(3x^2 - 5x + 4)$.

Solution We have

$$\begin{aligned}\lim_{x \to 2}(3x^2 - 5x + 4) &= \lim_{x \to 2}(3x^2) - \lim_{x \to 2}(5x) + \lim_{x \to 2} 4 && \text{By Theorem 3.1 [illegible]}\\ &= 3\lim_{x \to 2} x^2 - 5\lim_{x \to 2} x + 4 && \text{By Theorem 3.1 [illegible]}\\ &= 3 \cdot (2)^2 - 5 \cdot 2 + 4 = 6. && \text{By [illegible]}\end{aligned}$$

EXAMPLE 3.2 Finding the Limit of a Rational Function

Apply the rules of limits to evaluate $\lim_{x \to 3} \frac{x^3 - 5x + 4}{x^2 - 2}$.

Solution We get

$$\begin{aligned}\lim_{x \to 3} \frac{x^3 - 5x + 4}{x^2 - 2} &= \frac{\lim_{x \to 3}(x^3 - 5x + 4)}{\lim_{x \to 3}(x^2 - 2)} && \text{By Theorem 3.1 [illegible]}\\ &= \frac{\lim_{x \to 3} x^3 - 5\lim_{x \to 3} x + \lim_{x \to 3} 4}{\lim_{x \to 3} x^2 - \lim_{x \to 3} 2} && \text{By Theorem 3.1 [illegible]}\\ &= \frac{3^3 - 5 \cdot 3 + 4}{3^2 - 2} = \frac{16}{7}. && \text{By (3.4).}\end{aligned}$$

You may have noticed that in examples 3.1 and 3.2, we simply ended up substituting the value for x, after taking many intermediate steps. In example 3.3, it's not quite so simple.

EXAMPLE 3.3 Finding a Limit by Factoring

Evaluate $\lim_{x \to 1} \frac{x^2 - 1}{1 - x}$.

Solution Notice right away that

$$\lim_{x \to 1} \frac{x^2 - 1}{1 - x} \neq \frac{\lim_{x \to 1}(x^2 - 1)}{\lim_{x \to 1}(1 - x)},$$

since the limit in the denominator is zero. (Recall that the limit of a quotient is the quotient of the limits *only* when both limits exist *and* the limit in the denominator is *not* zero.) We can resolve this problem by observing that

$$\lim_{x\to1}\frac{x^2-1}{1-x}=\lim_{x\to1}\frac{(x-1)(x+1)}{-(x-1)}$$

[illegible]

$$=\lim_{x\to1}\frac{(x+1)}{-1}=-2,$$

Simplifying and [illegible]

where the cancellation of the factors of $(x-1)$ is valid because in the limit as $x\to1$, x is close to 1, but $x\neq1$, so that $x-1\neq0$. ■

In Theorem 3.2, we show that the limit of a polynomial is simply the value of the polynomial at that point; that is, to find the limit of a polynomial, we simply substitute in the value that x is approaching.

THEOREM 3.2

For any polynomial $p(x)$ and any real number a,

$$\lim_{x\to a}p(x)=p(a).$$

PROOF

Suppose that $p(x)$ is a polynomial of degree $n\geq0$,

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

Then, from Theorem 3.1 and (3.4),

$$\begin{aligned}\lim_{x\to a}p(x)&=\lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\&=c_n\lim_{x\to a}x^n+c_{n-1}\lim_{x\to a}x^{n-1}+\cdots+c_1\lim_{x\to a}x+\lim_{x\to a}c_0\\&=c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0=p(a).\ \blacksquare\end{aligned}$$

Evaluating the limit of a polynomial is now easy. Many other limits are evaluated just as easily.

THEOREM 3.3

Suppose that $\lim_{x\to a}f(x)=L$ and n is any positive integer. Then,

$$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}=\sqrt[n]{L},$$

where for n even, we assume that $L>0$.

The proof of Theorem 3.3 is given in Appendix A. Notice that this result says that we may (under the conditions outlined in the hypotheses) bring limits "inside" nth roots. We can then use our existing rules for computing the limit inside.

EXAMPLE 3.4 Evaluating the Limit of an nth Root of a Polynomial

Evaluate $\lim_{x\to2}\sqrt[5]{3x^2-2x}$.

Solution By Theorems 3.2 and 3.3, we have

$$\lim_{x\to2}\sqrt[5]{3x^2-2x}=\sqrt[5]{\lim_{x\to2}(3x^2-2x)}=\sqrt[5]{8}.\ ■$$

REMARK 3.3

In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation, as we do in examples 3.3 and 3.5.

EXAMPLE 3.5 Finding a Limit by Rationalizing

Evaluate $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$.

Solution First, notice that both the numerator $(\sqrt{x+2}-\sqrt{2})$ and the denominator (x) approach 0 as x approaches 0. Unlike example 3.3, we can't factor the numerator. However, we can rationalize the numerator, as follows:

$$\frac{\sqrt{x+2}-\sqrt{2}}{x} = \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})} = \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$
$$= \frac{x}{x(\sqrt{x+2}+\sqrt{2})} = \frac{1}{\sqrt{x+2}+\sqrt{2}},$$

where the last equality holds if $x \neq 0$ (which is the case in the limit as $x \to 0$). So, we have

$$\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

So that we are not restricted to discussing only the algebraic functions (i.e., those that can be constructed by using addition, subtraction, multiplication, division, exponentiation and by taking nth roots), we state the following result now, without proof.

THEOREM 3.4

For any real number a, we have

(i) $\lim_{x\to a} \sin x = \sin a$,

(ii) $\lim_{x\to a} \cos x = \cos a$,

(iii) $\lim_{x\to a} e^x = e^a$ and

(iv) $\lim_{x\to a} \ln x = \ln a$, for $a > 0$.

(v) $\lim_{x\to a} \sin^{-1} x = \sin^{-1} a$, for $-1 < a < 1$,

(vi) $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(vii) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a$, for $-\infty < a < \infty$ and

(viii) if p is a polynomial and $\lim_{x\to p(a)} f(x) = L$, then $\lim_{x\to a} f(p(x)) = L$.

Notice that Theorem 3.4 says that limits of the sine, cosine, exponential, natural logarithm, inverse sine, inverse cosine and inverse tangent functions are found simply by substitution. A more thorough discussion of functions with this property (called *continuity*) is found in section 1.4.

EXAMPLE 3.6 Evaluating a Limit of an Inverse Trigonometric Function

Evaluate $\lim_{x\to 0} \sin^{-1}\left(\frac{x+1}{2}\right)$.

Solution By Theorem 3.4, we have

$$\lim_{x\to 0} \sin^{-1}\left(\frac{x+1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

So much for limits that we can compute using elementary rules. Many limits can be found only by using more careful analysis, often by an indirect approach. For instance, consider the following problem.

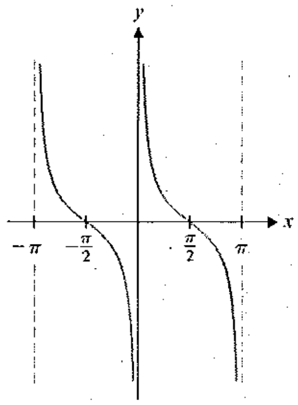

FIGURE 1.16
$y = \cot x$

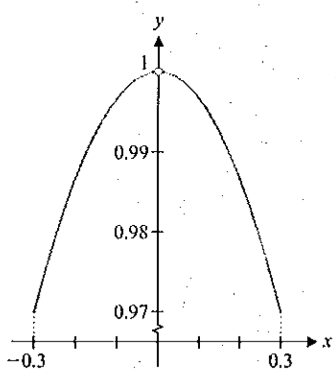

FIGURE 1.17
$y = x \cot x$

x	$x \cot x$
±0.1	0.9967
±0.01	0.999967
±0.001	0.99999967
±0.0001	0.9999999967
±0.00001	0.999999999967

EXAMPLE 3.7 A Limit of a Product That Is Not the Product of the Limits

Evaluate $\lim_{x\to 0}(x \cot x)$.

Solution Your first reaction might be to say that this is a limit of a product and so, must be the product of the limits:

$$\lim_{x\to 0}(x\cot x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\cot x\right) \quad \text{This is incorrect!}$$
$$= 0 \cdot ? = 0, \tag{3.5}$$

where we've written a "?" since you probably don't know what to do with $\lim_{x\to 0}\cot x$. Since the first limit is 0, do we really need to worry about the second limit? The problem here is that we are attempting to apply the result of Theorem 3.1 in a case where the hypotheses are not satisfied. Specifically, Theorem 3.1 says that the limit of a product is the product of the respective limits *when all of the limits exist.* The graph in Figure 1.16 suggests that $\lim_{x\to 0}\cot x$ does not exist. You should compute some function values, as well, to convince yourself that this is in fact the case. So, equation (3.5) does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph (see Figure 1.17) and compute some function values. Based on these, we conjecture that

$$\lim_{x\to 0}(x\cot x) = 1,$$

which is definitely not 0, as you might have initially suspected. You can also think about this limit as follows:

$$\lim_{x\to 0}(x\cot x) = \lim_{x\to 0}\left(x\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x}\cos x\right)$$
$$= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0}\cos x\right)$$
$$= \frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1,$$

since $\lim_{x\to 0}\cos x = 1$ and where we have used the conjecture we made in example 2.4 that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. (We verify this last conjecture in section 2.6, using the Squeeze Theorem, which follows.) ∎

At this point, we introduce a tool that will help us determine a number of important limits.

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x\to a} g(x) = L, \text{ also.}$$

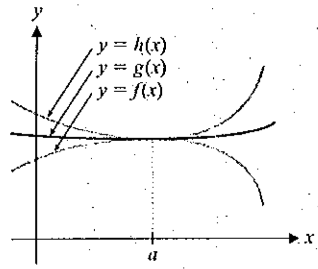

FIGURE 1.18
The Squeeze Theorem

REMARK 3.2

The Squeeze Theorem also applies to one-sided limits.

The proof of Theorem 3.5 is given in Appendix A, since it depends on the precise definition of limit found in section 1.6. However, if you refer to Figure 1.18, you should clearly see that if $g(x)$ lies between $f(x)$ and $h(x)$, except possibly at a itself and both $f(x)$ and $h(x)$ have the same limit as $x \to a$, then $g(x)$ gets *squeezed* between $f(x)$ and $h(x)$ and therefore should also have a limit of L. The challenge in using the Squeeze Theorem is in finding appropriate functions f and h that bound a given function g from below and above, respectively, and that have the same limit as $x \to a$.

EXAMPLE 3.8 Using the Squeeze Theorem to Verify the Value of a Limit

Determine the value of $\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right]$.

Solution Your first reaction might be to observe that this is a limit of a product and so, might be the product of the limits:

$$\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right] \stackrel{?}{=} \left(\lim_{x\to 0} x^2\right)\left[\lim_{x\to 0}\cos\left(\frac{1}{x}\right)\right]. \quad \text{This is incorrect!} \tag{3.6}$$

However, the graph of $y = \cos\left(\frac{1}{x}\right)$ found in Figure 1.19 suggests that $\cos\left(\frac{1}{x}\right)$ oscillates back and forth between -1 and 1. Further, the closer x gets to 0, the more rapid the oscillations become. You should compute some function values, as well, to convince yourself that $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist. Equation (3.6) then does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph and compute some function values in an effort to see what is going on. The graph of $y = x^2\cos\left(\frac{1}{x}\right)$ appears in Figure 1.20 and a table of function values is shown in the margin.

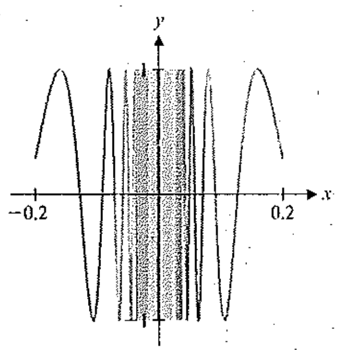

FIGURE 1.19
$y = \cos\left(\frac{1}{x}\right)$

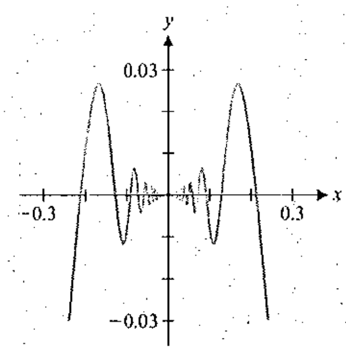

FIGURE 1.20
$y = x^2\cos\left(\frac{1}{x}\right)$

x	$x^2\cos(1/x)$
± 0.1	-0.008
± 0.01	8.6×10^{-5}
± 0.001	5.6×10^{-7}
± 0.0001	-9.5×10^{-9}
± 0.00001	-9.99×10^{-11}

The graph and the table of function values suggest the conjecture

$$\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right] = 0,$$

which we prove using the Squeeze Theorem. First, we need to find functions f and h such that

$$f(x) \le x^2\cos\left(\frac{1}{x}\right) \le h(x),$$

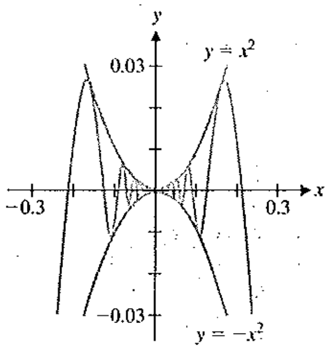

FIGURE 1.21
$y = x^2 \cos\left(\frac{1}{x}\right)$, $y = x^2$ and $y = -x^2$

for all $x \neq 0$ and where $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$. Recall that

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1, \tag{3.7}$$

for all $x \neq 0$. If we multiply (3.7) through by x^2 (notice that since $x^2 \geq 0$, this multiplication preserves the inequalities), we get

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2,$$

for all $x \neq 0$. We illustrate this inequality in Figure 1.21. Further,

$$\lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0} x^2.$$

So, from the Squeeze Theorem, it now follows that

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0,$$

also, as we had conjectured. ■

TODAY IN MATHEMATICS

Michael Freedman (1951–)
An American mathematician who first solved one of the most famous problems in mathematics, the four-dimensional Poincaré conjecture. A winner of the Fields Medal, the mathematical equivalent of the Nobel Prize, Freedman says, "Much of the power of mathematics comes from combining insights from seemingly different branches of the discipline. Mathematics is not so much a collection of different subjects as a way of thinking. As such, it may be applied to any branch of knowledge." Freedman finds mathematics to be an open field for research, saying that, "It isn't necessary to be an old hand in an area to make a contribution."

BEYOND FORMULAS

To resolve the limit in example 3.8, we could not apply the rules for limits contained in Theorem 3.1. So, we resorted to an indirect method of finding the limit. This tour de force of graphics plus calculation followed by analysis is sometimes referred to as the **Rule of Three.** (The Rule of Three presents a general strategy for attacking new problems. The basic idea is to look at problems graphically, numerically and analytically.) In the case of example 3.8, the first two elements of this "rule" (the graphics in Figure 1.20 and the accompanying table of function values) suggest a plausible conjecture, while the third element provides us with a careful mathematical verification of the conjecture. In what ways does this sound like the scientific method?

Functions are often defined by different expressions on different intervals. Such **piecewise-defined** functions are important and we illustrate such a function in example 3.9.

EXAMPLE 3.9 A Limit for a Piecewise-Defined Function

Evaluate $\lim_{x \to 0} f(x)$, where f is defined by

$$f(x) = \begin{cases} x^2 + 2\cos x + 1, & \text{for } x < 0 \\ e^x - 4, & \text{for } x \geq 0 \end{cases}.$$

Solution Since f is defined by different expressions for $x < 0$ and for $x \geq 0$, we must consider one-sided limits. We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 2\cos x + 1) = 2\cos 0 + 1 = 3,$$

by Theorem 3.4. Also, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (e^x - 4) = e^0 - 4 = 1 - 4 = -3.$$

Since the one-sided limits are different, we have that $\lim_{x \to 0} f(x)$ does not exist. ■

We end this section with an example of the use of limits in computing velocity. In section 2.1, we see that for an object moving in a straight line, whose position at time t is given by the function $f(t)$, the instantaneous velocity of that object at time $t = 1$ (i.e., the velocity at the *instant* $t = 1$, as opposed to the average velocity over some period of time) is given by the limit

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

EXAMPLE 3.10 Evaluating a Limit Describing Velocity

Suppose that the position function for an object at time t (seconds) is given by

$$f(t) = t^2 + 2 \text{ (feet)},$$

find the instantaneous velocity of the object at time $t = 1$.

Solution Given what we have just learned about limits, this is now an easy problem to solve. We have

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[(1+h)^2 + 2] - 3}{h}.$$

While we can't simply substitute $h = 0$ (why not?), we can write

$$\lim_{h \to 0} \frac{[(1+h)^2 + 2] - 3}{h} = \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h}$$ Expanding the squared term.

$$= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} \frac{2+h}{1} = 2.$$ Canceling factors of h

So, the instantaneous velocity of this object at time $t = 1$ is 2 feet per second. ■

EXERCISES 1.3

WRITING EXERCISES

1. Given your knowledge of the graphs of polynomials, explain why equations (3.1) and (3.2) and Theorem 3.2 are obvious. Name five non-polynomial functions for which limits can be evaluated by substitution.

2. Suppose that you can draw the graph of $y = f(x)$ without lifting your pencil from your paper. Explain why $\lim_{x \to a} f(x) = f(a)$, for every value of a.

3. In one or two sentences, explain the Squeeze Theorem. Use a real-world analogy (e.g., having the functions represent the locations of three people as they walk) to indicate why it is true.

4. Given the graph in Figure 1.20 and the calculations that follow, it may be unclear why we insist on using the Squeeze Theorem before concluding that $\lim_{x \to 0} [x^2 \cos(1/x)]$ is indeed 0. Review section 1.2 to explain why we are being so fussy.

In exercises 1–34, evaluate the indicated limit, if it exists. Assume that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

1. $\lim_{x \to 0} (x^2 - 3x + 1)$
2. $\lim_{x \to 2} \sqrt[3]{2x + 1}$
3. $\lim_{x \to 0} \cos^{-1}(x^2)$
4. $\lim_{x \to 2} \frac{x - 5}{x^2 + 4}$
5. $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$
6. $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$
7. $\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 4}$
8. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 + 2x - 3}$
9. $\lim_{x \to 0} \frac{\sin x}{\tan x}$
10. $\lim_{t \to 0} \frac{\tan t}{t}$
11. $\lim_{x \to 0} \frac{xe^{-2x+1}}{x^2 + x}$
12. $\lim_{t \to 0^+} x^2 \csc^2 x$

13. $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

14. $\lim_{x\to 0} \frac{2x}{3-\sqrt{x+9}}$

15. $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$

16. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

17. $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$

18. $\lim_{x\to 0} \left(\frac{2}{x} - \frac{2}{|x|}\right)$

19. $\lim_{x\to 0} \frac{1-e^{2x}}{1-e^x}$

20. $\lim_{x\to 0} \sin(e^{-1/x^2})$

21. $\lim_{x\to 0} \frac{\sin |x|}{x}$

22. $\lim_{x\to 0} \frac{\sin^2(x^2)}{x^4}$

23. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

24. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

25. $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2+2 & \text{if } x < -1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

26. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

27. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

28. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

29. $\lim_{h\to 0} \frac{(2+h)^2-4}{h}$

30. $\lim_{h\to 0} \frac{(1+h)^3-1}{h}$

31. $\lim_{h\to 0} \frac{h^2}{\sqrt{h^2+h+3}-\sqrt{h+3}}$

32. $\lim_{x\to 0} \frac{\sqrt{x^2+x+4}-2}{x^2+x}$

33. $\lim_{t\to -2} \frac{\frac{1}{2}+\frac{1}{t}}{2+t}$

34. $\lim_{x\to 0} \frac{\tan 2x}{5x}$

35. Use numerical and graphical evidence to conjecture the value of $\lim_{x\to 0} x^2 \sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions f and h, show graphically that $f(x) \le x^2 \sin(1/x) \le h(x)$ and justify $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x)$.

36. Why can't you use the Squeeze Theorem as in exercise 35 to prove that $\lim_{x\to 0} x^2 \sec(1/x) = 0$? Explore this limit graphically.

37. Use the Squeeze Theorem to prove that $\lim_{x\to 0^+} [\sqrt{x}\cos^2(1/x)] = 0$. Identify the functions f and h, show graphically that $f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$ for all $x > 0$, and justify $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^+} h(x) = 0$.

38. Suppose that $f(x)$ is bounded: that is, there exists a constant M such that $|f(x)| \le M$ for all x. Use the Squeeze Theorem to prove that $\lim_{x\to 0} x^2 f(x) = 0$.

In exercises 39–42, either find the limit or explain why it does not exist.

39. $\lim_{x\to 4^+} \sqrt{16-x^2}$

40. $\lim_{x\to 4^-} \sqrt{16-x^2}$

41. $\lim_{x\to -2^-} \sqrt{x^2+3x+2}$

42. $\lim_{x\to -2^+} \sqrt{x^2+3x+2}$

43. Given that $\lim_{x\to 0^+} \frac{1-\cos x}{x^2} = \frac{1}{2}$, quickly evaluate $\lim_{x\to 0^+} \frac{\sqrt{1-\cos x}}{x}$.

44. Given that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, quickly evaluate $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2}$.

45. Suppose $f(x) = \begin{cases} g(x) & \text{if } x < a \\ h(x) & \text{if } x > a \end{cases}$ for polynomials $g(x)$ and $h(x)$. Explain why $\lim_{x\to a^-} f(x) = g(a)$ and determine $\lim_{x\to a^+} f(x)$.

46. Explain how to determine $\lim_{x\to a} f(x)$ if g and h are polynomials and $f(x) = \begin{cases} g(x) & \text{if } x < a \\ c & \text{if } x = a \\ h(x) & \text{if } x > a \end{cases}$.

47. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim_{x\to 2}(x^2-3x+1)$ (b) $\lim_{x\to 0} \frac{x-2}{x^2+1}$

48. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim_{x\to -1} [(x+1)\sin x]$ (b) $\lim_{x\to 1} \frac{xe^x}{\tan x}$

In exercises 49–52, use the given position function $f(t)$ to find the velocity at time $t = a$.

49. $f(t) = t^2+2, a = 2$

50. $f(t) = t^2+2, a = 0$

51. $f(t) = t^3, a = 0$

52. $f(t) = t^3, a = 1$

53. In Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is defined by $m = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$. Compute the slope m. Graph $y = \sqrt{x}$ and the line with slope m through the point $(1, 1)$.

54. In Chapter 2, an alternative form for the limit in exercise 53 is given by $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$. Compute this limit.

In exercises 55–62, use numerical evidence to conjecture the value of the limit if it exists. Check your answer with your Computer Algebra System (CAS). If you disagree, which one of you is correct?

55. $\lim_{x\to 0^-} (1+x)^{1/x}$

56. $\lim_{x\to 0^+} e^{1/x}$

57. $\lim_{x\to 0^+} x^{-x^2}$

58. $\lim_{x\to 0^-} x^{\ln x}$

59. $\lim_{x\to 0} \sin \frac{1}{x}$

60. $\lim_{x\to 0} e^{1/x}$

61. $\lim_{x\to 0} \tan^{-1} \frac{1}{x}$

62. $\lim_{x\to 0} \ln \left|\frac{1}{x}\right|$

In exercises 63–66, use $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -3$ and $\lim_{x\to a} h(x) = 0$ to determine the limit, if possible.

63. $\lim_{x\to a}[2f(x) - 3g(x)]$

64. $\lim_{x\to a}[3f(x)g(x)]$

65. $\lim_{x\to a} \left[\frac{f(x)+g(x)}{h(x)}\right]$

66. $\lim_{x\to a} \left[\frac{3f(x)+2g(x)}{h(x)}\right]$

67. Assume that $\lim_{x\to a} f(x) = L$. Use Theorem 3.1 to prove that $\lim_{x\to a}[f(x)]^3 = L^3$. Also, show that $\lim_{x\to a}[f(x)]^4 = L^4$.

68. How did you work exercise 67? You probably used Theorem 3.1 to work from $\lim_{x\to a}[f(x)]^2 = L^2$ to $\lim_{x\to a}[f(x)]^3 = L^3$, and then used $\lim_{x\to a}[f(x)]^3 = L^3$ to get $\lim_{x\to a}[f(x)]^4 = L^4$. Going one step at a time, we should be able to reach $\lim_{x\to a}[f(x)]^n = L^n$ for any positive integer n. This is the idea of **mathematical induction.** Formally, we need to show the result is true for a specific value of $n = n_0$ [we show $n_0 = 2$ in the text], then assume the result is true for a general $n = k \geq n_0$. If we show that we can get from the result being true for $n = k$ to the result being true for $n = k + 1$, we have proved that the result is true for any positive integer n. In one sentence, explain why this is true. Use this technique to prove that $\lim_{x\to a}[f(x)]^n = L^n$ for any positive integer n.

69. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{1}{x} = \lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0} x \lim_{x\to 0}\frac{1}{x^2} = 0\cdot ? = 0.$$

70. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \frac{0}{0} = 1.$$

71. Give an example of functions f and g such that $\lim_{x\to 0}[f(x) + g(x)]$ exists but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

72. Give an example of functions f and g such that $\lim_{x\to 0}[f(x)\cdot g(x)]$ exists but at least one of $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

73. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, is it always true that $\lim_{x\to a}[f(x) + g(x)]$ does not exist? Explain.

74. Is the following true or false? If $\lim_{x\to 0} f(x)$ does not exist, then $\lim_{x\to 0}\frac{1}{f(x)}$ does not exist. Explain.

75. Suppose a state's income tax code states the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0.14x & \text{if } 0 \leq x < 10{,}000 \\ 1500 + 0.21x & \text{if } 10{,}000 \leq x \end{cases}.$$

Compute $\lim_{x\to 0^+} T(x)$; why is this good? Compute $\lim_{x\to 10{,}000} T(x)$; why is this bad?

76. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} a + 0.12x & \text{if } x \leq 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $\lim_{x\to 0^+} T(x) = 0$ and $\lim_{x\to 20{,}000} T(x)$ exists. Why is it important for these limits to exist?

77. The **greatest integer function** is denoted by $f(x) = [x]$ and equals the greatest integer that is less than or equal to x. Thus, $[2.3] = 2$, $[-1.2] = -2$ and $[3] = 3$. In spite of this last fact, show that $\lim_{x\to 3}[x]$ does not exist.

78. Investigate the existence of (a) $\lim_{x\to 1}[x]$, (b) $\lim_{x\to 1.5}[x]$, (c) $\lim_{x\to 1.5}[2x]$, and (d) $\lim_{x\to 1}(x - [x])$.

EXPLORATORY EXERCISES

1. The value $x = 0$ is called a **zero of multiplicity** n $(n \geq 1)$ for the function f if $\lim_{x\to 0}\frac{f(x)}{x^n}$ exists and is nonzero but $\lim_{x\to 0}\frac{f(x)}{x^{n-1}} = 0$. Show that $x = 0$ is a zero of multiplicity 2 for x^2, $x = 0$ is a zero of multiplicity 3 for x^3 and $x = 0$ is a zero of multiplicity 4 for x^4. For polynomials, what does multiplicity describe? The reason the definition is not as straightforward as we might like is so that it can apply to non-polynomial functions, as well. Find the multiplicity of $x = 0$ for $f(x) = \sin x$; $f(x) = x\sin x$; $f(x) = \sin x^2$. If you know that $x = 0$ is a zero of multiplicity m for $f(x)$ and multiplicity n for $g(x)$, what can you say about the multiplicity of $x = 0$ for $f(x) + g(x)$? $f(x)\cdot g(x)$? $f(g(x))$?

2. We have conjectured that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Using graphical and numerical evidence, conjecture the value of $\lim_{x\to 0}\frac{\sin 2x}{x}$, $\lim_{x\to 0}\frac{\sin 3x}{x}$, $\lim_{x\to 0}\frac{\sin \pi x}{x}$ and $\lim_{x\to 0}\frac{\sin x/2}{x}$. In general, conjecture the value of $\lim_{x\to 0}\frac{\sin cx}{x}$ for any constant c. Given that $\lim_{x\to 0}\frac{\sin cx}{cx} = 1$ for any constant $c \neq 0$, prove that your conjecture is correct.

1.4 CONTINUITY AND ITS CONSEQUENCES

When you describe something as *continuous,* just what do you have in mind? For example, if told that a machine has been in *continuous* operation for the past 60 hours, most of us would interpret this to mean that the machine has been in operation *all* of that time, without

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When you describe something as *continuous,* just what do you have in mind? For example, if told that a machine has been in *continuous* operation for the past 60 hours, most of us would interpret this to mean that the machine has been in operation *all* of that time, without any interruption at all, even for a moment. Mathematicians mean much the same thing when we say that a function is continuous. A function is said to be *continuous* on an interval if its graph on that interval can be drawn without interruption, that is, without lifting your pencil from the paper.

It is helpful for us to first try to see what it is about the functions whose graphs are shown in Figures 1.22a–1.22d that makes them *discontinuous* (i.e., not continuous) at the point $x = a$.

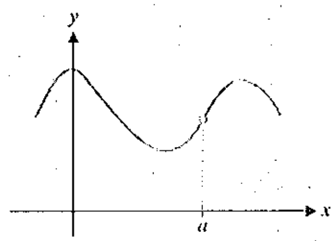


FIGURE 1.22a
$f(a)$ is not defined (the graph has a hole at $x = a$).

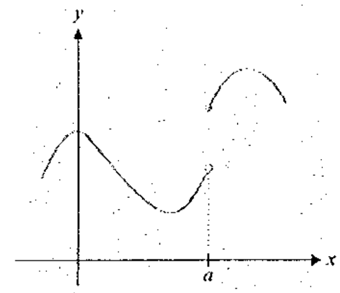


FIGURE 1.22b
$f(a)$ is defined, but $\lim_{x \to a} f(x)$ does not exist (the graph has a jump at $x = a$).

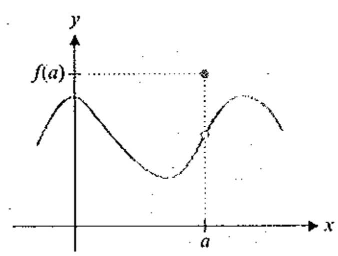


FIGURE 1.22c
$\lim_{x \to a} f(x)$ exists and $f(a)$ is defined, but $\lim_{x \to a} f(x) \neq f(a)$ (the graph has a hole at $x = a$).

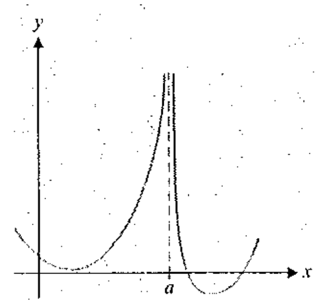


FIGURE 1.22d
$\lim_{x \to a} f(x)$ does not exist (the function "blows up" at $x = a$).

REMARK 4.1

The definition of continuity all boils down to the one condition in (iii), since conditions (i) and (ii) must hold whenever (iii) is met. Further, this says that a function is continuous at a point exactly when you can compute its limit at that point by simply substituting in.

This suggests the following definition of continuity at a point.

DEFINITION 4.1

A function f is **continuous** at $x = a$ when
(i) $f(a)$ is defined, (ii) $\lim_{x \to a} f(x)$ exists and (iii) $\lim_{x \to a} f(x) = f(a)$.
Otherwise, f is said to be **discontinuous** at $x = a$.

For most purposes, it is best for you to think of the intuitive notion of continuity that we've outlined above. Definition 4.1 should then simply follow from your intuitive understanding of the concept.

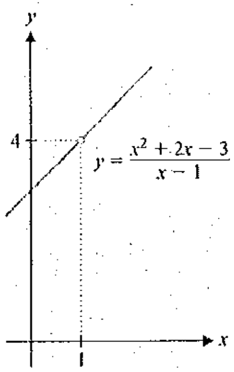


FIGURE 1.23
$y = \dfrac{x^2+2x-3}{x-1}$

EXAMPLE 4.1 Finding Where a Rational Function Is Continuous

Determine where $f(x) = \dfrac{x^2+2x-3}{x-1}$ is continuous.

Solution Note that

$$\begin{aligned} f(x) &= \frac{x^2+2x-3}{x-1} = \frac{(x-1)(x+3)}{x-1} && \text{Factoring the numerator} \\ &= x+3, \text{ for } x \neq 1. && \text{Canceling common factors.} \end{aligned}$$

This says that the graph of f is a straight line, but with a hole in it at $x = 1$, as indicated in Figure 1.23. So, f is discontinuous at $x = 1$, but continuous elsewhere. ■

EXAMPLE 4.2 Removing a Discontinuity

Make the function from example 4.1 continuous everywhere by redefining it at a single point.

Solution In example 4.1, we saw that the function is discontinuous at $x = 1$, since it is undefined there. So, suppose we just go ahead and define it, as follows. Let

$$g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1}, & \text{if } x \neq 1 \\ a, & \text{if } x = 1, \end{cases}$$

for some real number a.

Notice that $g(x)$ is defined for all x and equals $f(x)$ for all $x \neq 1$. Here, we have

$$\begin{aligned} \lim_{x\to 1} g(x) &= \lim_{x\to 1} \frac{x^2+2x-3}{x-1} \\ &= \lim_{x\to 1} (x+3) = 4. \end{aligned}$$

Observe that if we choose $a = 4$, we now have that

$$\lim_{x\to 1} g(x) = 4 = g(1)$$

and so, g is continuous at $x = 1$.

Note that the graph of g is the same as the graph of f seen in Figure 1.23, except that we now include the point (1, 4) (see Figure 1.24). Also, note that there's a very simple way to write $g(x)$. (Think about this.) ■

REMARK 4.2

You should be careful not to confuse the continuity of a function at a point with its simply being defined there. A function can be defined at a point without being continuous there. (Look back at Figures 1.22b and 1.22c.)

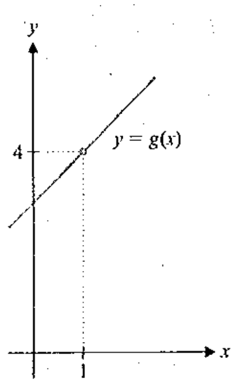


FIGURE 1.24
$y = g(x)$

When we can remove a discontinuity by redefining the function at that point, we call the discontinuity **removable.** Not all discontinuities are removable, however. Carefully examine Figures 1.22a–1.22d and convince yourself that the discontinuities in Figures 1.22a and 1.22c are removable, while those in Figures 1.22b and 1.22d are nonremovable. Briefly, a function f has a removable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists and either $f(a)$ is undefined or $\lim_{x\to a} f(x) \neq f(a)$.

EXAMPLE 4.3 Nonremovable Discontinuities

Find all discontinuities of $f(x) = \dfrac{1}{x^2}$ and $g(x) = \cos\left(\dfrac{1}{x}\right)$.

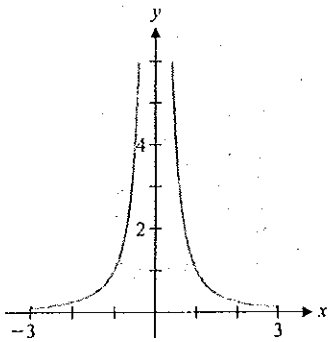


FIGURE 1.25a
$y = \frac{1}{x^2}$

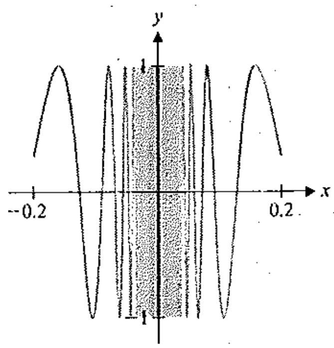


FIGURE 1.25b
$y = \cos(1/x)$

Solution You should observe from Figure 1.25a (also, construct a table of function values) that

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist.}$$

Hence, f is discontinuous at $x = 0$.

Similarly, observe that $\lim_{x\to 0} \cos(1/x)$ does not exist, due to the endless oscillation of $\cos(1/x)$ as x approaches 0 (see Figure 1.25b).

In both cases, notice that since the limits do not exist, there is no way to redefine either function at $x = 0$ to make it continuous there. ■

From your experience with the graphs of some common functions, the following result should come as no surprise.

THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1} x$ and e^x are continuous everywhere, $\sqrt[n]{x}$ is continuous for all x, when n is odd and for $x > 0$, when n is even. We also have $\ln x$ is continuous for $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$.

PROOF

We have already established (in Theorem 3.2) that for any polynomial $p(x)$ and any real number a,

$$\lim_{x\to a} p(x) = p(a),$$

from which it follows that p is continuous at $x = a$. The rest of the theorem follows from Theorem 3.4 in a similar way. ■

From these very basic continuous functions, we can build a large collection of continuous functions, using Theorem 4.2.

THEOREM 4.2

Suppose that f and g are continuous at $x = a$. Then all of the following are true:

(i) $(f \pm g)$ is continuous at $x = a$,
(ii) $(f \cdot g)$ is continuous at $x = a$ and
(iii) (f/g) is continuous at $x = a$ if $g(a) \neq 0$.

Simply put, Theorem 4.2 says that a sum, difference or product of continuous functions is continuous, while the quotient of two continuous functions is continuous at any point at which the denominator is nonzero.

PROOF

(i) If f and g are continuous at $x = a$, then

$$\begin{aligned}\lim_{x\to a}[f(x) \pm g(x)] &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) && \text{From Theorem 3.1}\\ &= f(a) \pm g(a) && \text{Since } f \text{ and } g \text{ are continuous at } a.\\ &= (f \pm g)(a),\end{aligned}$$

by the usual rules of limits. Thus, $(f \pm g)$ is also continuous at $x = a$.

Parts (ii) and (iii) are proved in a similar way and are left as exercises. ■

EXAMPLE 4.4 Continuity for a Rational Function

Determine where f is continuous, for $f(x) = \dfrac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$.

Solution Here, f is a quotient of two polynomial (hence continuous) functions. The graph of the function indicated in Figure 1.26 suggests a vertical asymptote at around $x = 4$, but doesn't indicate any other discontinuity. From Theorem 4.2, f will be continuous at all x where the denominator is not zero, that is, where

$$x^2 - 3x - 4 = (x + 1)(x - 4) \neq 0.$$

Thus, f is continuous for $x \neq -1, 4$. (Think about why you didn't see anything peculiar about the graph at $x = -1$.) ■

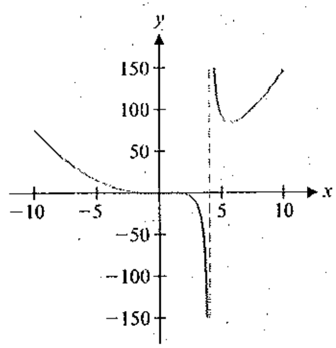


FIGURE 1.26
$y = \dfrac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$

With the addition of the result in Theorem 4.3, we will have all the basic tools needed to establish the continuity of most elementary functions.

THEOREM 4.3

Suppose that $\lim_{x\to a} g(x) = L$ and f is continuous at L. Then,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

A proof of Theorem 4.3 is given in Appendix A.

Notice that this says that if f is continuous, then we can bring the limit "inside." This should make sense, since as $x \to a$, $g(x) \to L$ and so, $f(g(x)) \to f(L)$, since f is continuous at L.

COROLLARY 4.1

Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a.

PROOF

From Theorem 4.3, we have

$$\lim_{x\to a}(f \circ g)(x) = \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$
$$= f(g(a)) = (f \circ g)(a). \quad \text{Since } g \text{ is continuous at } a. \blacksquare$$

EXAMPLE 4.5 Continuity for a Composite Function

Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

Solution Note that

$$h(x) = f(g(x)),$$

where $g(x) = x^2 - 5x + 2$ and $f(x) = \cos x$. Since both f and g are continuous for all x, h is continuous for all x, by Corollary 4.1. ■

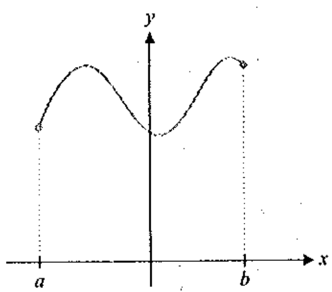


FIGURE 1.27
f continuous on $[a, b]$

DEFINITION 4.2

If f is continuous at every point on an open interval (a, b), we say that f is **continuous on (a, b).** Following Figure 1.27, we say that f **is continuous on the closed interval $[a, b]$,** if f is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is **continuous.** (That is, when we don't specify an interval, we mean continuous everywhere.)

For many functions, it's a simple matter to determine the intervals on which the function is continuous. We illustrate this in example 4.6.

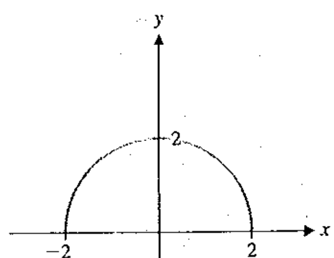


FIGURE 1.28
$y = \sqrt{4 - x^2}$

EXAMPLE 4.6 Continuity on a Closed Interval

Determine the interval(s) where f is continuous, for $f(x) = \sqrt{4 - x^2}$.

Solution First, observe that f is defined only for $-2 \le x \le 2$. Next, note that f is the composition of two continuous functions and hence, is continuous for all x for which $4 - x^2 > 0$. We show a graph of the function in Figure 1.28. Since

$$4 - x^2 > 0$$

for $-2 < x < 2$, we have that f is continuous for all x in the interval $(-2, 2)$, by Theorem 4.1 and Corollary 4.1. Finally, we test the endpoints to see that $\lim_{x\to 2^-} \sqrt{4 - x^2} = 0 = f(2)$ and $\lim_{x\to -2^+} \sqrt{4 - x^2} = 0 = f(-2)$, so that f is continuous on the closed interval $[-2, 2]$. ■

EXAMPLE 4.7 Interval of Continuity for a Logarithm

Determine the interval(s) where $f(x) = \ln(x - 3)$ is continuous.

Solution It follows from Theorem 4.1 and Corollary 4.1 that f is continuous whenever $(x - 3) > 0$ (i.e., for $x > 3$). Thus, f is continuous on the interval $(3, \infty)$. ■

The Internal Revenue Service presides over some of the most despised functions in existence. Look up the current Tax Rate Schedules. In 2002, the first few lines (for single taxpayers) looked like:

For taxable amount over	*but not over*	*your tax liability is*	*minus*
\$0	\$6000	10%	\$0
\$6000	\$27,950	15%	\$300
\$27,950	\$ 67,700	27%	\$3654

Where do the numbers \$300 and \$3654 come from? If we write the tax liability $T(x)$ as a function of the taxable amount x (assuming that x can be any real value and not just a whole dollar amount), we have

$$T(x) = \begin{cases} 0.10x & \text{if } 0 < x \le 6000 \\ 0.15x - 300 & \text{if } 6000 < x \le 27{,}950 \\ 0.27x - 3654 & \text{if } 27{,}950 < x \le 67{,}700. \end{cases}$$

Be sure you understand our translation so far. Note that it is important that this be a continuous function: think of the fairness issues that would arise if it were not!

EXAMPLE 4.8 Continuity of Federal Tax Tables

Verify that the federal tax rate function $T(x)$ is continuous at the "joint" $x = 27{,}950$. Then, find a to complete the table. (You will find b and c as exercises.)

For taxable amount over	*but not over*	*your tax liability is*	*minus*
\$67,700	\$141,250	30%	a
\$141,250	\$307,050	35%	b
\$307,050	—	38.6%	c

Solution For $T(x)$ to be continuous at $x = 27{,}950$, we must have

$$\lim_{x\to 27{,}950^-} T(x) = \lim_{x\to 27{,}950^+} T(x).$$

Since both functions $0.15x - 300$ and $0.27x - 3654$ are continuous, we can compute the one-sided limits by substituting $x = 27{,}950$. Thus,

$$\lim_{x\to 27{,}950^-} T(x) = 0.15(27{,}950) - 300 = 3892.50$$

and

$$\lim_{x\to 27{,}950^+} T(x) = 0.27(27{,}950) - 3654 = 3892.50.$$

Since the one-sided limits agree and equal the value of the function at that point, $T(x)$ is continuous at $x = 27{,}950$. We leave it as an exercise to establish that $T(x)$ is also continuous at $x = 6000$. (It's worth noting that the function could be written with equal signs on all of the inequalities; this would be incorrect if the function were discontinuous.) To complete the table, we choose a to get the one-sided limits at $x = 67{,}700$ to match. We have

$$\lim_{x\to 67{,}700^-} T(x) = 0.27(67{,}700) - 3654 = 14{,}625,$$

while $$\lim_{x\to 67,700^+} T(x) = 0.30(67,700) - a = 20,310 - a.$$

So, we set the one-sided limits equal, to obtain

$$14,625 = 20,310 - a$$

or $$a = 20,310 - 14,625 = 5685.$$

Theorem 4.4 should seem an obvious consequence of our intuitive definition of continuity.

HISTORICAL NOTES

Karl Weierstrass (1815–1897) A German mathematician who proved the Intermediate Value Theorem and several other fundamental results of the calculus. Weierstrass was known as an excellent teacher whose students circulated his lecture notes throughout Europe, because of their clarity and originality. Also known as a superb fencer, Weierstrass was one of the founders of modern mathematical analysis.

THEOREM 4.4 (Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.

Theorem 4.4 says that if f is continuous on $[a, b]$, then f must take on *every* value between $f(a)$ and $f(b)$ at least once. That is, a continuous function cannot skip over any numbers between its values at the two endpoints. To do so, the graph would need to leap across the horizontal line $y = W$, something that continuous functions cannot do (see Figure 1.29a). Of course, a function may take on a given value W more than once (see Figure 1.29b). We must point out that, although these graphs make this result seem reasonable, like any other result, Theorem 4.4 requires proof. The proof is more complicated than you might imagine and we must refer you to an advanced calculus text.

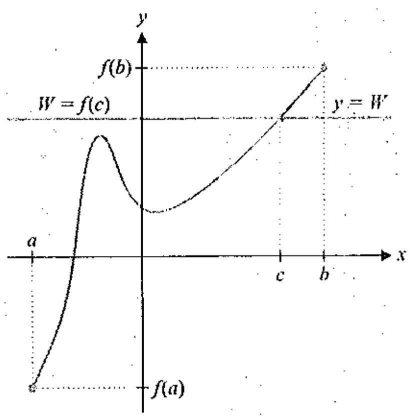


FIGURE 1.29a
An illustration of the Intermediate Value Theorem

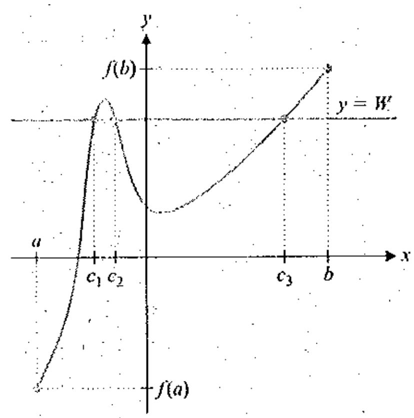


FIGURE 1.29b
More than one value of c

In Corollary 4.2, we see an immediate and useful application of the Intermediate Value Theorem.

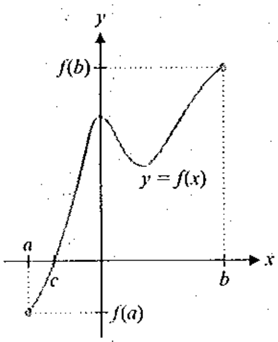


FIGURE 1.30
Intermediate Value Theorem where c is a zero of f

COROLLARY 4.2

Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs [i.e., $f(a) \cdot f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$. (Recall that c is then a *zero* of f.)

Notice that Corollary 4.2 is simply the special case of the Intermediate Value Theorem where $W = 0$ (see Figure 1.30). The Intermediate Value Theorem and Corollary 4.2 are examples of *existence theorems;* they tell you that there *exists* a number c satisfying some condition, but they do *not* tell you what c is.

○ The Method of Bisections

In example 4.9, we see how Corollary 4.2 can help us locate the zeros of a function.

EXAMPLE 4.9 Finding Zeros by the Method of Bisections

Find the zeros of $f(x) = x^5 + 4x^2 - 9x + 3$.

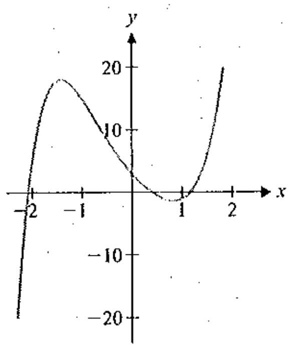


FIGURE 1.31
$y = x^5 + 4x^2 - 9x + 3$

Solution If f were a quadratic polynomial, you could certainly find its zeros. However, you don't have any formulas for finding zeros of polynomials of degree 5. The only alternative is to approximate the zeros. A good starting place would be to draw a graph of $y = f(x)$ like the one in Figure 1.31. There are three zeros visible on the graph. Since f is a polynomial, it is continuous everywhere and so, Corollary 4.2 says that there must be a zero on any interval on which the function changes sign. From the graph, you can see that there must be zeros between -3 and -2, between 0 and 1 and between 1 and 2. We could also conclude this by computing say, $f(0) = 3$ and $f(1) = -1$. Although we've now found intervals that contain zeros, the question remains as to how we can *find* the zeros themselves.

While a rootfinding program can provide an accurate approximation, the issue here is not so much to get an answer as it is to understand how to find one. We suggest a simple yet effective method, called the **method of bisections.**

For the zero between 0 and 1, a reasonable guess might be the midpoint, 0.5. Since $f(0.5) \approx -0.469 < 0$ and $f(0) = 3 > 0$, there must be a zero between 0 and 0.5. Next, the midpoint of $[0, 0.5]$ is 0.25 and $f(0.25) \approx 1.001 > 0$, so that the zero is on the interval $(0.25, 0.5)$. We continue in this way to narrow the interval on which there's a zero until the interval is sufficiently small so that any point in the interval can serve as an adequate approximation to the actual zero. We do this in the following table.

a	*b*	*f(a)*	*f(b)*	*Midpoint*	*f(midpoint)*
0	1	3	−1	0.5	−0.469
0	0.5	3	−0.469	0.25	1.001
0.25	0.5	1.001	−0.469	0.375	0.195
0.375	0.5	0.195	−0.469	0.4375	−0.156
0.375	0.4375	0.195	−0.156	0.40625	0.015
0.40625	0.4375	0.015	−0.156	0.421875	−0.072
0.40625	0.421875	0.015	−0.072	0.4140625	−0.029
0.40625	0.4140625	0.015	−0.029	0.41015625	−0.007
0.40625	0.41015625	0.015	−0.007	0.408203125	0.004

If you continue this process through 20 more steps, you ultimately arrive at the approximate zero $x = 0.40892288$, which is accurate to at least eight decimal places. ■

This method of bisections is a tedious process, if you're working it with pencil and paper. It is interesting because it's reliable and it's a simple, yet general method for finding approximate zeros. Computer and calculator rootfinding utilities are very useful, but our purpose here is to provide you with an understanding of how basic rootfinding works. We discuss a more powerful method for finding roots in Chapter 3.

EXERCISES 1.4

WRITING EXERCISES

1. Think about the following "real-life" functions, each of which is a function of the independent variable time: the height of a falling object, the velocity of an object, the amount of money in a bank account, the cholesterol level of a person, the heart rate of a person, the amount of a certain chemical present in a test tube and a machine's most recent measurement of the cholesterol level of a person. Which of these are continuous functions? For each function you identify as discontinuous, what is the real-life meaning of the discontinuities?

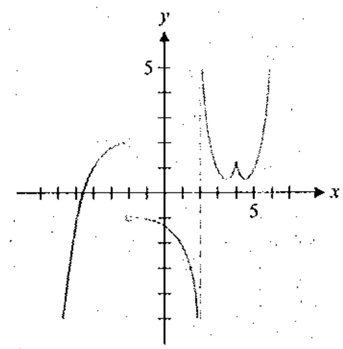
2. Whether a process is continuous or not is not always clear-cut. When you watch television or a movie, the action seems to be continuous. This is an optical illusion, since both movies and television consist of individual "snapshots" that are played back at many frames per second. Where does the illusion of continuous motion come from? Given that the average person blinks several times per minute, is our perception of the world actually continuous? (In what cognitive psychologists call **temporal binding**, the human brain first decides whether a stimulus is important enough to merit conscious consideration. If so, the brain "predates" the stimulus so that the person correctly identifies when the stimulus actually occurred.)

3. When you sketch the graph of the parabola $y = x^2$ with pencil or pen, is your sketch (at the molecular level) actually the graph of a continuous function? Is your calculator or computer's graph actually the graph of a continuous function? On many calculators, you have the option of a connected or disconnected graph. At the pixel level, does a connected graph show the graph of a function? Does a disconnected graph show the graph of a continuous function? Do we ever have problems correctly interpreting a graph due to these limitations? In the exercises in section 1.7, we examine one case where our perception of a computer graph depends on which choice is made.

4. For each of the graphs in Figures 1.22a–1.22d, describe (with an example) what the formula for $f(x)$ might look like to produce the given discontinuity.

In exercises 1–6, use the given graph to identify all discontinuities of the functions.

1.



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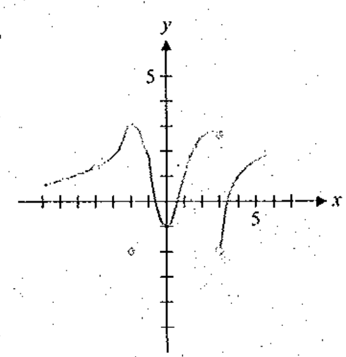

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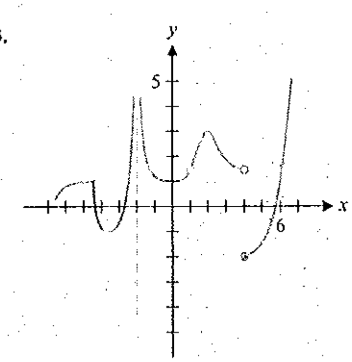
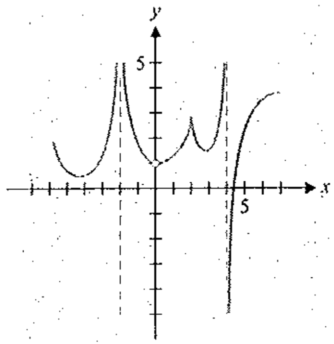


4.



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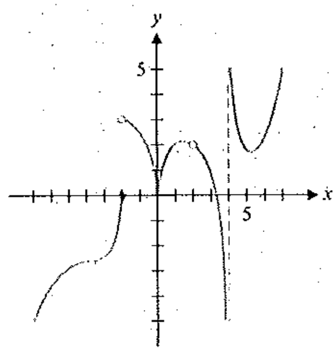

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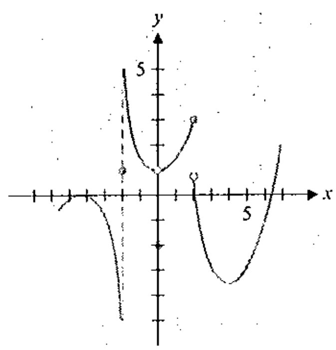


In exercises 7–12, explain why each function is discontinuous at the given point by indicating which of the three conditions in Definition 4.1 are not met.

7. $f(x) = \dfrac{x}{x-1}$ at $x = 1$

8. $f(x) = \dfrac{x^2-1}{x-1}$ at $x = 1$

9. $f(x) = \sin\dfrac{1}{x}$ at $x = 0$

10. $f(x) = e^{1/x}$ at $x = 0$

11. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$ at $x = 2$

12. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$ at $x = 2$

In exercises 13–24, find all discontinuities of $f(x)$. For each discontinuity that is removable, define a new function that removes the discontinuity.

13. $f(x) = \dfrac{x-1}{x^2-1}$

14. $f(x) = \dfrac{4x}{x^2+x-2}$

15. $f(x) = \dfrac{4x}{x^2+4}$

16. $f(x) = \dfrac{3x}{x^2-2x-4}$

17. $f(x) = x^2 \tan x$

18. $f(x) = x \cot x$

19. $f(x) = x \ln x^2$

20. $f(x) = e^{-4/x^2}$

21. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

23. $f(x) = \begin{cases} 3x-1 & \text{if } x \leq -1 \\ x^2+5x & \text{if } -1 < x < 1 \\ 3x^3 & \text{if } x \geq 1 \end{cases}$

24. $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ \sin x & \text{if } 0 < x \leq \pi \\ x-\pi & \text{if } x > \pi \end{cases}$

In exercises 25–32, determine the intervals on which $f(x)$ is continuous.

25. $f(x) = \sqrt{x+3}$

26. $f(x) = \sqrt{x^2-4}$

27. $f(x) = \sqrt[3]{x+2}$

28. $f(x) = (x-1)^{3/2}$

29. $f(x) = \sin(x^2+2)$

30. $f(x) = \cos\left(\dfrac{1}{x}\right)$

31. $f(x) = \ln(x+1)$

32. $f(x) = \ln(4-x^2)$

In exercises 33–35, determine values of a and b that make the given function continuous.

33. $f(x) = \begin{cases} \dfrac{2\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos x & \text{if } x > 0 \end{cases}$

34. $f(x) = \begin{cases} ae^x + 1 & \text{if } x < 0 \\ \sin^{-1}\dfrac{x}{2} & \text{if } 0 \leq x \leq 2 \\ x^2 - x + b & \text{if } x > 2 \end{cases}$

35. $f(x) = \begin{cases} a(\tan^{-1} x + 2) & \text{if } x < 0 \\ 2e^{bx} + 1 & \text{if } 0 \leq x \leq 3 \\ \ln(x-2) + x^2 & \text{if } x > 3 \end{cases}$

36. Prove Corollary 4.1.

37. Suppose that a state's income tax code states that the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 0.14x & \text{if } 0 < x < 10{,}000 \\ c + 0.21x & \text{if } 10{,}000 \leq x. \end{cases}$$

Determine the constant c that makes this function continuous for all x. Give a rationale why such a function should be continuous.

38. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} 0 & \text{if } x \le 0 \\ a + 0.12x & \text{if } 0 < x \le 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $T(x)$ is continuous for all x.

39. In example 4.8, find b and c to complete the table.

40. In example 4.8, show that $T(x)$ is continuous for $x = 6000$.

In exercises 41–46, use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

41. $f(x) = x^2 - 7$, $[2, 3]$

42. $f(x) = x^3 - 4x - 2$, $[2, 3]$

43. $f(x) = x^3 - 4x - 2$, $[-1, 0]$

44. $f(x) = x^3 - 4x - 2$, $[-2, -1]$

45. $f(x) = \cos x - x$, $[0, 1]$

46. $f(x) = e^x + x$, $[-1, 0]$

A function is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$. In exercises 47–50, determine whether $f(x)$ is continuous from the right at $x = 2$.

47. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 1 & \text{if } x \ge 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

49. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

50. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$

51. Define what it means for a function to be **continuous from the left** at $x = a$ and determine which of the functions in exercises 47–50 are continuous from the left at $x = 2$.

52. Suppose that $f(x) = \dfrac{g(x)}{h(x)}$ and $h(a) = 0$. Determine whether each of the following statements is always true, always false, or maybe true/maybe false. Explain. (a) $\lim_{x \to a} f(x)$ does not exist. (b) $f(x)$ is discontinuous at $x = a$.

53. The sex of newborn Mississippi alligators is determined by the temperature of the eggs in the nest. The eggs fail to develop unless the temperature is between 26°C and 36°C. All eggs between 26°C and 30°C develop into females, and eggs between 34°C and 36°C develop into males. The percentage of females decreases from 100% at 30°C to 0% at 34°C. If $f(T)$ is the percentage of females developing from an egg at T°C, then

$$f(T) = \begin{cases} 100 & \text{if } 26 \le T \le 30 \\ g(T) & \text{if } 30 < T < 34 \\ 0 & \text{if } 34 \le T \le 36, \end{cases}$$

for some function $g(T)$. Explain why it is reasonable that $f(T)$ be continuous. Determine a function $g(T)$ such that $0 \le g(T) \le 100$ for $30 \le T \le 34$ and the resulting function $f(T)$ is continuous. [Hint: It may help to draw a graph first and make $g(T)$ linear.]

54. If $f(x) = \begin{cases} x^2, & \text{if } x \ne 0 \\ 4, & \text{if } x = 0 \end{cases}$ and $g(x) = 2x$, show that

$$\lim_{x \to 0} f(g(x)) \ne f\left(\lim_{x \to 0} g(x)\right).$$

55. If you push on a large box resting on the ground, at first nothing will happen because of the static friction force that opposes motion. If you push hard enough, the box will start sliding, although there is again a friction force that opposes the motion. Suppose you are given the following description of the friction force. Up to 100 pounds, friction matches the force you apply to the box. Over 100 pounds, the box will move and the friction force will equal 80 pounds. Sketch a graph of friction as a function of your applied force based on this description. Where is this graph discontinuous? What is significant physically about this point? Do you think the friction force actually ought to be continuous? Modify the graph to make it continuous while still retaining most of the characteristics described.

56. For $f(x) = 2x - \dfrac{400}{x}$, we have $f(-1) > 0$ and $f(2) < 0$. Does the Intermediate Value Theorem guarantee a zero of $f(x)$ between $x = -1$ and $x = 2$? What happens if you try the method of bisections?

57. On Monday morning, a saleswoman leaves on a business trip at 7:13 A.M. and arrives at her destination at 2:03 P.M. The following morning, she leaves for home at 7:17 A.M. and arrives at 1:59 P.M. The woman notices that at a particular stoplight along the way, a nearby bank clock changes from 10:32 A.M. to 10:33 A.M. on both days. Therefore, she must have been at the same location at the same time on both days. Her boss doesn't believe that such an unlikely coincidence could occur. Use the Intermediate Value Theorem to argue that it *must* be true that at some point on the trip, the saleswoman was at exactly the same place at the same time on both Monday and Tuesday.

58. Suppose you ease your car up to a stop sign at the top of a hill. Your car rolls back a couple of feet and then you drive through the intersection. A police officer pulls you over for not coming to a complete stop. Use the Intermediate Value Theorem to argue that there was an instant in time when your car was stopped (in fact, there were at least two). What is the difference between this stopping and the stopping that the police officer wanted to see?

59. Suppose a worker's salary starts at \$40,000 with \$2000 raises every 3 months. Graph the salary function $s(t)$; why is it discontinuous? How does the function $f(t) = 40{,}000 + \frac{2000}{3}t$ (t in months) compare? Why might it be easier to do calculations with $f(t)$ than $s(t)$?

60. Prove the final two parts of Theorem 4.2.

61. Suppose that $f(x)$ is a continuous function with consecutive zeros at $x = a$ and $x = b$; that is, $f(a) = f(b) = 0$ and $f(x) \neq 0$ for $a < x < b$. Further, suppose that $f(c) > 0$ for some number c between a and b. Use the Intermediate Value Theorem to argue that $f(x) > 0$ for all $a < x < b$.

62. Use the method of bisections to estimate the other two zeros in example 4.9.

63. Suppose that $f(x)$ is continuous at $x = 0$. Prove that $\lim_{x\to 0} xf(x) = 0$.

64. The **converse** of exercise 63 is not true. That is, the fact $\lim_{x\to 0} xf(x) = 0$ does not guarantee that $f(x)$ is continuous at $x = 0$. Find a counterexample; that is, find a function f such that $\lim_{x\to 0} xf(x) = 0$ and $f(x)$ is not continuous at $x = 0$.

65. If $f(x)$ is continuous at $x = a$, prove that $g(x) = |f(x)|$ is continuous at $x = a$.

66. Determine whether the converse of exercise 65 is true. That is, if $|f(x)|$ is continuous at $x = a$, is it necessarily true that $f(x)$ must be continuous at $x = a$?

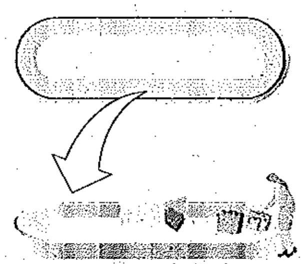
67. Let $f(x)$ be a continuous function for $x \geq a$ and define $h(x) = \max_{a \leq t \leq x} f(t)$. Prove that $h(x)$ is continuous for $x \geq a$. Would this still be true without the assumption that $f(x)$ is continuous?

68. Graph $f(x) = \dfrac{\sin|x^3 - 3x^2 + 2x|}{x^3 - 3x^2 + 2x}$ and determine all discontinuities.

EXPLORATORY EXERCISES

1. In the text, we discussed the use of the method of bisections to find an approximate solution of equations such as $f(x) = x^3 + 5x - 1 = 0$. We can start by noticing that $f(0) = -1$ and $f(1) = 5$. Since $f(x)$ is continuous, the Intermediate Value Theorem tells us that there is a solution between $x = 0$ and $x = 1$. For the method of bisections, we guess the midpoint, $x = 0.5$. Is there any reason to suspect that the solution is actually closer to $x = 0$ than to $x = 1$? Using the function values $f(0) = -1$ and $f(1) = 5$, devise your own method of guessing the location of the solution. Generalize your method to using $f(a)$ and $f(b)$, where one function value is positive and one is negative. Compare your method to the method of bisections on the problem $x^3 + 5x - 1 = 0$; for both methods, stop when you are within 0.001 of the solution, $x \approx 0.198437$. Which method performed better? Before you get overconfident in your method, compare the two methods again on $x^3 + 5x^2 - 1 = 0$. Does your method get close on the first try? See if you can determine graphically why your method works better on the first problem.

2. You have probably seen the turntables on which luggage rotates at the airport. Suppose that such a turntable has two long straight parts with a semicircle on each end (see the figure). We will model the left/right movement of the luggage. Suppose the straight part is 40 ft long, extending from $x = -20$ to $x = 20$. Assume that our luggage starts at time $t = 0$ at location $x = -20$, and that it takes 60 s for the luggage to reach $x = 20$. Suppose the radius of the circular portion is 5 ft and it takes the luggage 30 s to complete the half-circle. We model the straight-line motion with a linear function $x(t) = at + b$. Find constants a and b so that $x(0) = -20$ and $x(60) = 20$. For the circular motion, we use a cosine (Why is this a good choice?) $x(t) = 20 + d \cdot \cos(et + f)$ for constants d, e and f. The requirements are $x(60) = 20$ (since the motion is continuous), $x(75) = 25$ and $x(90) = 20$. Find values of d, e and f to make this work. Find equations for the position of the luggage along the backstretch and the other semicircle. What would the motion be from then on?



Luggage carousel

3. Determine all x's for which each function is continuous.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases}$$

$$g(x) = \begin{cases} x^2 + 3 & \text{if } x \text{ is irrational} \\ 4x & \text{if } x \text{ is rational and} \end{cases}$$

$$h(x) = \begin{cases} \cos 4x & \text{if } x \text{ is irrational} \\ \sin 4x & \text{if } x \text{ is rational} \end{cases}$$

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

In this section, we revisit some old limit problems to give more informative answers and examine some related questions.

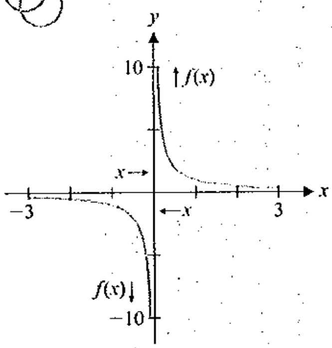


FIGURE 1.32

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10,000
0.00001	100,000

x	$\frac{1}{x}$
−0.1	−10
−0.01	−100
−0.001	−1000
−0.0001	−10,000
−0.00001	−100,000

EXAMPLE 5.1 A Simple Limit Revisited

Examine $\lim_{x\to 0} \frac{1}{x}$.

Solution Of course, we can draw a graph (see Figure 1.32) and compute a table of function values easily, by hand. (See the tables in the margin.)

While we say that the limits $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$ do not exist, the behavior of the function is clearly quite different for $x > 0$ than for $x < 0$. Specifically, as $x \to 0^+$, $\frac{1}{x}$ increases without bound, while as $x \to 0^-$, $\frac{1}{x}$ decreases without bound. To communicate more about the behavior of the function near $x = 0$, we write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \tag{5.1}$$

and

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty. \tag{5.2}$$

Graphically, this says that the graph of $y = \frac{1}{x}$ approaches the vertical line $x = 0$, as $x \to 0$, as seen in Figure 1.32. When this occurs, we say that the line $x = 0$ is a **vertical asymptote.** It is important to note that while the limits (5.1) and (5.2) *do not exist*, we say that they "equal" ∞ and $-\infty$, respectively, only to be specific as to *why* they do not exist. Finally, in view of the one-sided limits (5.1) and (5.2), we say that

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

REMARK 5.1

It may at first seem contradictory to say that $\lim_{x\to 0^+} \frac{1}{x}$ does not exist and then to write $\lim_{x\to 0^+} \frac{1}{x} = \infty$. Note that since ∞ is *not* a real number, there is no contradiction here. (When we say that a limit "does not exist," we are saying that there is no real number L that the function values are approaching.) We say that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ to indicate that as $x \to 0^+$, the function values are increasing without bound.

EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite

Evaluate $\lim_{x\to 0} \frac{1}{x^2}$.

Solution The graph (in Figure 1.33) seems to indicate a vertical asymptote at $x = 0$. A table of values is easily constructed by hand (see the accompanying tables).

x	$\frac{1}{x^2}$
0.1	100
0.01	10,000
0.001	1×10^6
0.0001	1×10^8
0.00001	1×10^{10}

x	$\frac{1}{x^2}$
−0.1	100
−0.01	10,000
−0.001	1×10^6
−0.0001	1×10^8
−0.00001	1×10^{10}

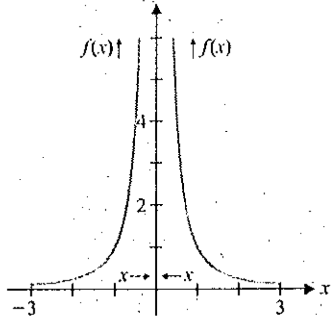


FIGURE 1.33
$\lim_{x\to 0}\frac{1}{x^2}=\infty$

From this, we can see that

$$\lim_{x\to 0^+}\frac{1}{x^2}=\infty$$

and
$$\lim_{x\to 0^-}\frac{1}{x^2}=\infty.$$

Since both one-sided limits agree (i.e., both tend to ∞), we say that

$$\lim_{x\to 0}\frac{1}{x^2}=\infty.$$

This one concise statement says that the limit does not exist, but also that $f(x)$ has a vertical asymptote at $x=0$ where $f(x)\to\infty$ as $x\to 0$ from either side. ■

REMARK 5.2

Mathematicians try to convey as much information as possible with as few symbols as possible. For instance, we prefer to say $\lim_{x\to 0}\frac{1}{x^2}=\infty$ rather than $\lim_{x\to 0}\frac{1}{x^2}$ does not exist, since the first statement not only says that the limit does not exist, but also says that $\frac{1}{x^2}$ increases without bound as x approaches 0, with $x>0$ or $x<0$.

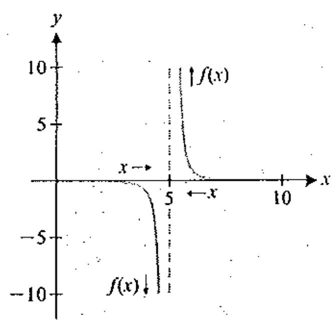


FIGURE 1.34
$\lim_{x\to 5^+}\frac{1}{(x-5)^3}=\infty$ and
$\lim_{x\to 5^-}\frac{1}{(x-5)^3}=-\infty$

EXAMPLE 5.3 A Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x\to 5}\frac{1}{(x-5)^3}$.

Solution In Figure 1.34, we show a graph of the function. From the graph, you should get a pretty clear idea that there's a vertical asymptote at $x=5$ and just how the function is blowing up there (to ∞ from the right side and to $-\infty$ from the left). You can verify this behavior algebraically, by noticing that as $x\to 5$, the denominator approaches 0, while the numerator approaches 1. This says that the fraction grows large in absolute value, without bound as $x\to 5$. Specifically,

$$\text{as } x\to 5^+,\quad (x-5)^3\to 0\quad\text{and}\quad (x-5)^3>0.$$

We indicate the sign of each factor by printing a small "+" or "−" sign above or below each one. This enables you to see the signs of the various terms at a glance. In this case, we have

$$\lim_{x\to 5^+}\frac{\overset{+}{1}}{\underset{+}{(x-5)^3}}=\infty.\qquad \text{Since } (x-5)^3>0 \text{, for } x>5.$$

Likewise, as $x\to 5^-$, $(x-5)^3\to 0$ and $(x-5)^3<0$.

In this case, we have

$$\lim_{x\to 5^-}\frac{\overset{+}{1}}{\underset{-}{(x-5)^3}}=-\infty.\qquad \text{Since } (x-5)^3<0 \text{, for } x<5.$$

Finally, we say that $\lim_{x\to 5}\frac{1}{(x-5)^3}$ does not exist,

since the one-sided limits are different. ■

Learning from the lessons of examples 5.1, 5.2 and 5.3, you should recognize that if the denominator tends to 0 and the numerator does not, then the limit in question does not exist. In this event, we can determine whether the limit tends to ∞ or $-\infty$ by carefully examining the signs of the various factors.

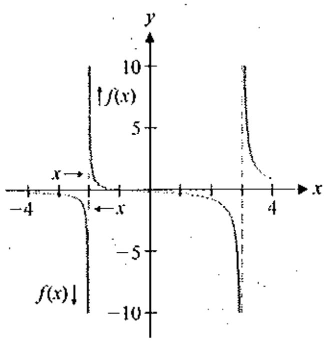


FIGURE 1.35
$\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)}$ does not exist.

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)}$.

Solution First, notice from the graph of the function shown in Figure 1.35 that there appears to be a vertical asymptote at $x = -2$.

Further, the function appears to tend to ∞ as $x \to -2^+$, and to $-\infty$ as $x \to -2^-$. You can verify this behavior, by observing that

$$\lim_{x\to -2^+} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{+}{(x+2)}} = \infty$$

and

$$\lim_{x\to -2^-} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{-}{(x+2)}} = -\infty.$$

So, we can see that $x = -2$ is indeed a vertical asymptote and that

$$\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)} \text{ does not exist.}$$

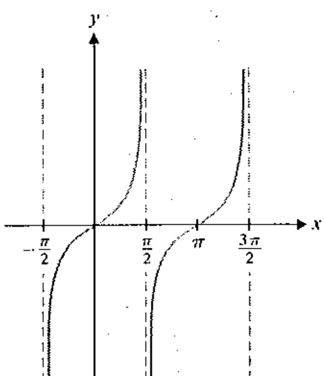


FIGURE 1.36
$y = \tan x$

EXAMPLE 5.5 A Limit Involving a Trigonometric Function

Evaluate $\lim_{x\to \frac{\pi}{2}} \tan x$.

Solution Notice from the graph of the function shown in Figure 1.36 that there appears to be a vertical asymptote at $x = \frac{\pi}{2}$.

You can verify this behavior by observing that

$$\lim_{x\to \frac{\pi}{2}^-} \tan x = \lim_{x\to \frac{\pi}{2}^-} \frac{\overset{+}{\sin x}}{\underset{+}{\cos x}} = \infty$$

and

$$\lim_{x\to \frac{\pi}{2}^+} \tan x = \lim_{x\to \frac{\pi}{2}^+} \frac{\overset{+}{\sin x}}{\underset{-}{\cos x}} = -\infty.$$

So, we can see that $x = \frac{\pi}{2}$ is indeed a vertical asymptote and that

$$\lim_{x\to \frac{\pi}{2}} \tan x \text{ does not exist.}$$

○ Limits at Infinity

We are also interested in examining the limiting behavior of functions as x increases without bound (written $x \to \infty$) or as x decreases without bound (written $x \to -\infty$).

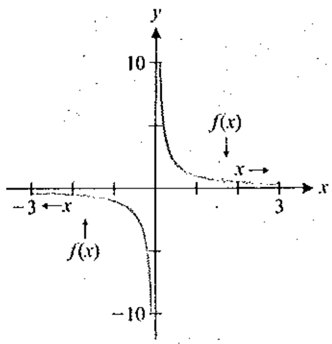


FIGURE 1.37
$\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$

Returning to $f(x) = \frac{1}{x}$, we can see that as $x \to \infty$, $\frac{1}{x} \to 0$. In view of this, we write

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Notice that in Figure 1.37, the graph appears to approach the horizontal line $y = 0$, as $x \to \infty$ and as $x \to -\infty$. In this case, we call $y = 0$ a **horizontal asymptote.**

EXAMPLE 5.6 Finding Horizontal Asymptotes

Look for any horizontal asymptotes of $f(x) = 2 - \frac{1}{x}$.

Solution We show a graph of $y = f(x)$ in Figure 1.38. Since as $x \to \pm\infty$, $\frac{1}{x} \to 0$, we get that

$$\lim_{x\to\infty} \left(2 - \frac{1}{x}\right) = 2$$

and

$$\lim_{x\to-\infty} \left(2 - \frac{1}{x}\right) = 2.$$

Thus, the line $y = 2$ is a horizontal asymptote. ◀

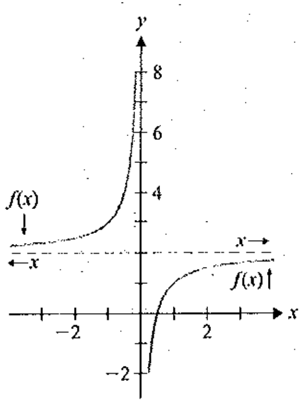


FIGURE 1.38
$\lim_{x\to\infty} \left(2 - \frac{1}{x}\right) = 2$ and
$\lim_{x\to-\infty} \left(2 - \frac{1}{x}\right) = 2$

As you can see in Theorem 5.1, the behavior of $\frac{1}{x^t}$, for any positive rational power t, as $x \to \pm\infty$, is largely the same as we observed for $f(x) = \frac{1}{x}$.

THEOREM 5.1

For any rational number $t > 0$,

$$\lim_{x\to\pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = \frac{p}{q}$ where q is odd.

REMARK 5.3

All of the usual rules for limits stated in Theorem 3.1 also hold for limits as $x \to \pm\infty$.

A proof of Theorem 5.1 is given in Appendix A. Be sure that the following argument makes sense to you: for $t > 0$, as $x \to \infty$, we also have $x^t \to \infty$, so that $\frac{1}{x^t} \to 0$.

In Theorem 5.2, we see that the behavior of a polynomial at infinity is easy to determine.

THEOREM 5.2

For a polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}.$$

PROOF

We have
$$\lim_{x\to\infty} p_n(x) = \lim_{x\to\infty}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0)$$
$$= \lim_{x\to\infty}\left[x^n\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)\right]$$
$$= \infty,$$
if $a_n > 0$, since
$$\lim_{x\to\infty}\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) = a_n$$
and $\lim_{x\to\infty} x^n = \infty$. The result is proved similarly for $a_n < 0$. ■

Observe that you can make similar statements regarding the value of $\lim_{x\to-\infty} p_n(x)$, but be careful: the answer will change depending on whether n is even or odd. (We leave this as an exercise.)

In example 5.7, we again see the need for caution when applying our basic rules for limits (Theorem 3.1), which also apply to limits as $x \to \infty$ or as $x \to -\infty$.

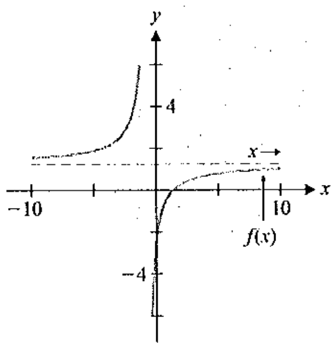


FIGURE 1.39
$$\lim_{x\to\infty}\frac{5x-7}{4x+3} = \frac{5}{4}$$

x	$\frac{5x-7}{4x+3}$
10	1
100	1.223325
1000	1.247315
10,000	1.249731
100,000	1.249973

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Evaluate $\lim_{x\to\infty}\dfrac{5x-7}{4x+3}$.

Solution You might be tempted to write
$$\lim_{x\to\infty}\frac{5x-7}{4x+3} = \frac{\lim_{x\to\infty}(5x-7)}{\lim_{x\to\infty}(4x+3)}$$
This is an incorrect use of Theorem 3.1 (iv), since the limits in the numerator and the denominator do not exist.
$$= \frac{\infty}{\infty} = 1. \qquad (5.3)$$
This is incorrect!

The graph in Figure 1.39 and some function values (see the accompanying table) suggest that the conjectured value of 1 is incorrect. Recall that the limit of a quotient is the quotient of the limits only when *both* limits exist (and the limit in the denominator is nonzero). Since both the limit in the denominator and that in the numerator tend to ∞, the limits *do not exist.*

Further, when a limit looks like $\frac{\infty}{\infty}$, the actual value of the limit can be anything at all. For this reason, we call $\frac{\infty}{\infty}$ an **indeterminate form,** meaning that the value of the expression cannot be determined solely by noticing that both numerator and denominator tend to ∞.

Rule of Thumb: When faced with the indeterminate form $\frac{\infty}{\infty}$ in calculating the limit of a rational function, divide numerator and denominator by the highest power of x appearing in the *denominator.*

Here, we have
$$\lim_{x\to\infty}\frac{5x-7}{4x+3} = \lim_{x\to\infty}\left[\frac{5x-7}{4x+3}\cdot\frac{(1/x)}{(1/x)}\right]$$
Multiply numerator and denominator by $\frac{1}{x}$.
$$= \lim_{x\to\infty}\frac{5-7/x}{4+3/x}$$
Multiply through by $\frac{1}{x}$.
$$= \frac{\lim_{x\to\infty}(5-7/x)}{\lim_{x\to\infty}(4+3/x)}$$
By Theorem 3.1 (iv).
$$= \frac{5}{4} = 1.25,$$
which is consistent with what we observed both graphically and numerically earlier. ■

In example 5.8, we apply our rule of thumb to a common limit problem.

EXAMPLE 5.8 Finding Slant Asymptotes

Evaluate $\lim_{x\to\infty} \frac{4x^3+5}{-6x^2-7x}$ and find any slant asymptotes.

Solution As usual, we first examine a graph (see Figure 1.40a). Note that here, the graph appears to tend to $-\infty$ as $x \to \infty$. Further, observe that outside of the interval $[-2, 2]$, the graph looks very much like a straight line. If we look at the graph in a somewhat larger window, this linearity is even more apparent (see Figure 1.40b).

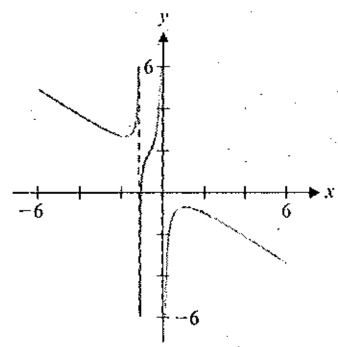


FIGURE 1.40a
$y = \frac{4x^3+5}{-6x^2-7x}$

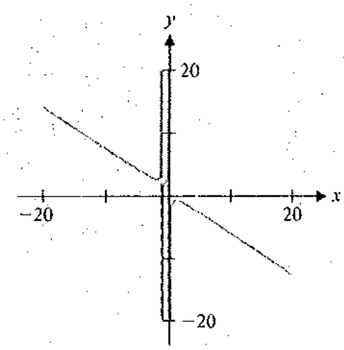


FIGURE 1.40b
$y = \frac{4x^3+5}{-6x^2-7x}$

Using our rule of thumb, we have

$$\lim_{x\to\infty} \frac{4x^3+5}{-6x^2-7x} = \lim_{x\to\infty}\left[\frac{4x^3+5}{-6x^2-7x}\cdot\frac{(1/x^2)}{(1/x^2)}\right] \quad \text{Multiply numerator and denominator by } \frac{1}{x^2}.$$

$$= \lim_{x\to\infty} \frac{4x+5/x^2}{-6-7/x} \quad \text{Multiply through by } \frac{1}{x^2}.$$

$$= -\infty,$$

since as $x \to \infty$, the numerator tends to ∞ and the denominator tends to -6.

To further explain the behavior seen in Figure 1.40b, we perform a long division. We have

$$\frac{4x^3+5}{-6x^2-7x} = -\frac{2}{3}x + \frac{7}{9} + \frac{5+49/9x}{-6x^2-7x}.$$

Since the third term in this expansion tends to 0 as $x \to \infty$, the function values approach those of the linear function

$$-\frac{2}{3}x + \frac{7}{9},$$

as $x \to \infty$. For this reason, we say that the function has a **slant (or oblique) asymptote.** That is, instead of approaching a vertical or horizontal line, as happens with vertical or horizontal asymptotes, the graph is approaching the slanted straight line $y = -\frac{2}{3}x + \frac{7}{9}$. (This is the behavior we're seeing in Figure 1.40b.) ■

Limits involving exponential functions are very important in many applications.

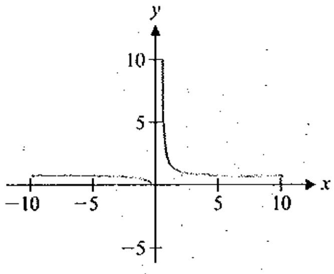


FIGURE 1.41a
$y = e^{1/x}$

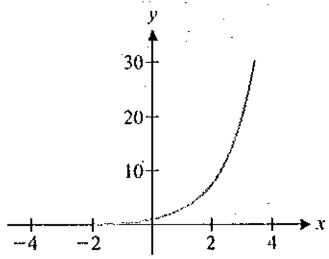


FIGURE 1.41b
$y = e^x$

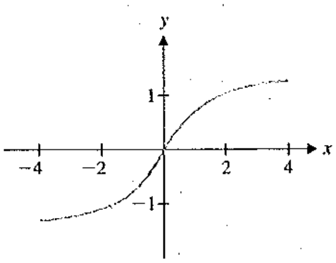


FIGURE 1.42a
$y = \tan^{-1} x$

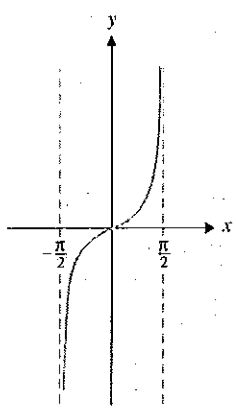


FIGURE 1.42b
$y = \tan x$

EXAMPLE 5.9 Two Limits of an Exponential Function

Evaluate $\lim_{x\to 0^-} e^{1/x}$ and $\lim_{x\to 0^+} e^{1/x}$.

Solution A computer-generated graph is shown in Figure 1.41a. Although it is an unusual looking graph, it appears that the function values are approaching 0, as x approaches 0 from the left and tend to infinity as x approaches 0 from the right. To verify this, recall that $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to -\infty} e^x = 0$. (See Figure 1.41b for a graph of $y = e^x$.) Combining these results, we get

$$\lim_{x\to 0^-} e^{1/x} = 0.$$

Similarly, $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to \infty} e^x = \infty$. (Again, see Figure 1.41b.) In this case, we have

$$\lim_{x\to 0^+} e^{1/x} = \infty. \blacksquare$$

As we see in example 5.10, inverse trigonometric functions may have horizontal asymptotes.

EXAMPLE 5.10 Two Limits of an Inverse Trigonometric Function

Evaluate $\lim_{x\to\infty} \tan^{-1} x$ and $\lim_{x\to -\infty} \tan^{-1} x$.

Solution The graph of $y = \tan^{-1} x$ (shown in Figure 1.42a) suggests a horizontal asymptote of about $y = -1.5$ as $x \to -\infty$ and about $y = 1.5$ as $x \to \infty$. We can be more precise with this, as follows. For $\lim_{x\to\infty} \tan^{-1} x$, we are looking for the angle that θ must approach, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that $\tan\theta$ tends to ∞. Referring to the graph of $y = \tan x$ in Figure 1.42b, we see that $\tan x$ tends to ∞ as x approaches $\frac{\pi}{2}^-$. Likewise, $\tan x$ tends to $-\infty$ as x approaches $-\frac{\pi}{2}^+$, so that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}. \blacksquare$$

In example 5.11, we consider a model of the size of an animal's pupils. Recall that in bright light, pupils shrink to reduce the amount of light entering the eye, while in dim light, pupils dilate to allow in more light. (See the chapter introduction.)

EXAMPLE 5.11 Finding the Size of an Animal's Pupils

Suppose that the diameter of an animal's pupils is given by $f(x)$ mm, where x is the intensity of light on the pupils. If $f(x) = \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15}$, find the diameter of the pupils with (a) minimum light and (b) maximum light.

Solution For part (a), notice that $f(0)$ is undefined, since $0^{-0.4}$ indicates a division by 0. We therefore consider the limit of $f(x)$ as x approaches 0, but we compute a one-sided limit since x cannot be negative. A computer-generated graph of $y = f(x)$ with $0 \le x \le 10$ is shown in Figure 1.43a. It appears that the y-values approach 20 as x approaches 0. To compute the limit, we multiply numerator and denominator by $x^{0.4}$ (to eliminate the negative exponents). We then have

$$\lim_{x\to 0^+} \frac{160x^{-0.4}+90}{4x^{-0.4}+15} = \lim_{x\to 0^+} \frac{160x^{-0.4}+90}{4x^{-0.4}+15} \cdot \frac{x^{0.4}}{x^{0.4}}$$
$$= \lim_{x\to 0^+} \frac{160+90x^{0.4}}{4+15x^{0.4}} = \frac{160}{4} = 40 \text{ mm}.$$

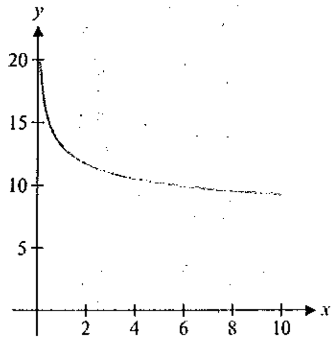


FIGURE 1.43a
$y = f(x)$

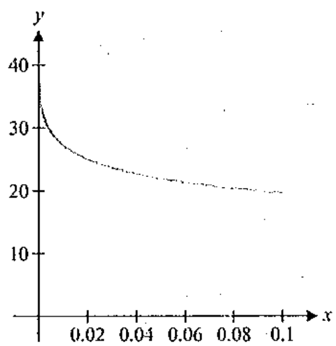


FIGURE 1.43b
$y = f(x)$

This limit does not seem to match our graph, but notice that Figure 1.43a shows a gap near $x = 0$. In Figure 1.43b, we have zoomed in so that $0 \le x \le 0.1$. Here, a limit of 40 looks more reasonable.

For part (b), we consider the limit as x tends to ∞. From Figure 1.43a, it appears that the graph has a horizontal asymptote at a value close to $y = 10$. We compute the limit

$$\lim_{x\to\infty} \frac{160x^{-0.4}+90}{4x^{-0.4}+15} = \frac{90}{15} = 6 \text{ mm}.$$

So, the pupils have a limiting size of 6 mm, as the intensity of light tends to ∞. ■

In our final example, we consider the velocity of a falling object.

EXAMPLE 5.12 Finding the Limiting Velocity of a Falling Object

The velocity in ft/s of a falling object is modeled by

$$v(t) = -\sqrt{\frac{32}{k}}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}},$$

where k is a constant that depends upon the size and shape of the object and the density of the air. Find the **limiting velocity** of the object, that is, find $\lim_{t\to\infty} v(t)$ and compare limiting velocities for skydivers with $k = 0.00016$ (head first) and $k = 0.001$ (spread eagle).

Solution Observe that the only place that t appears in the expression for $v(t)$ is in the two identical exponential terms: $e^{-2t\sqrt{32k}}$. Also notice that $\lim_{t\to\infty} e^{-2t\sqrt{32k}} = 0$, since $\lim_{x\to-\infty} e^x = 0$. We then have

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -\sqrt{\frac{32}{k}}\left(\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}\right)$$
$$= -\sqrt{\frac{32}{k}}\left(\frac{1-\lim_{t\to\infty} e^{-2t\sqrt{32k}}}{1+\lim_{t\to\infty} e^{-2t\sqrt{32k}}}\right) = -\sqrt{\frac{32}{k}}\left(\frac{1-0}{1+0}\right) = -\sqrt{\frac{32}{k}} \text{ ft/s},$$

where the negative sign indicates a downward direction. So, with $k = 0.00016$, the limiting velocity is $-\sqrt{\frac{32}{0.00016}} \approx -447$ ft/s (about 300 mph!), and with $k = 0.001$, the limiting velocity is $-\sqrt{\frac{32}{0.001}} \approx -179$ ft/s (about 122 mph). ■

EXERCISES 1.5

WRITING EXERCISES

1. It may seem odd that we use ∞ in describing limits but do not count ∞ as a real number. Discuss the existence of ∞: is it a number or a concept?

2. In example 5.7, we dealt with the "indeterminate form" $\frac{\infty}{\infty}$. Thinking of a limit of ∞ as meaning "getting very large" and a limit of 0 as meaning "getting very close to 0," explain why the following are indeterminate forms: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, and $\infty \cdot 0$. Determine what the following non-indeterminate forms represent: $\infty + \infty$, $-\infty - \infty$, $\infty + 0$ and $0/\infty$.

3. On your computer or calculator, graph $y = 1/(x-2)$ and look for the horizontal asymptote $y = 0$ and the vertical asymptote $x = 2$. Most computers will draw a vertical line at $x = 2$ and will show the graph completely flattening out at $y = 0$ for large x's. Is this accurate? misleading? Most computers will compute the locations of points for adjacent x's and try to connect the points with a line segment. Why might this result in a vertical line at the location of a vertical asymptote?

4. Many students learn that asymptotes are lines that the graph gets closer and closer to without ever reaching. This is true for many asymptotes, but not all. Explain why vertical asymptotes are never reached or crossed. Explain why horizontal or slant asymptotes may, in fact, be crossed any number of times; draw one example.

In exercises 1–4, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

1. (a) $\lim_{x\to 1^-} \frac{1-2x}{x^2-1}$ (b) $\lim_{x\to 1^+} \frac{1-2x}{x^2-1}$ (c) $\lim_{x\to 1} \frac{1-2x}{x^2-1}$

2. (a) $\lim_{x\to -1^-} \frac{1-2x}{x^2-1}$ (b) $\lim_{x\to -1^+} \frac{1-2x}{x^2-1}$ (c) $\lim_{x\to -1} \frac{1-2x}{x^2-1}$

3. (a) $\lim_{x\to 2^-} \frac{x-4}{x^2-4x+4}$ (b) $\lim_{x\to 2^+} \frac{x-4}{x^2-4x+4}$ (c) $\lim_{x\to 2} \frac{x-4}{x^2-4x+4}$

4. (a) $\lim_{x\to -1^-} \frac{1-x}{(x+1)^2}$ (b) $\lim_{x\to -1^+} \frac{1-x}{(x+1)^2}$ (c) $\lim_{x\to -1} \frac{1-x}{(x+1)^2}$

In exercises 5–24, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

5. $\lim_{x\to 2^-} \frac{-x}{\sqrt{4-x^2}}$

6. $\lim_{x\to -1^-} (x^2-2x-3)^{-2/3}$

7. $\lim_{x\to -\infty} \frac{-x}{\sqrt{4+x^2}}$

8. $\lim_{x\to \infty} \frac{2x^2-x+1}{4x^2-3x-1}$

9. $\lim_{x\to \infty} \frac{x^3-2}{3x^2+4x-1}$

10. $\lim_{x\to \infty} \frac{2x^2-1}{4x^3-5x-1}$

11. $\lim_{x\to \infty} \ln 2x$

12. $\lim_{x\to 0^+} \ln 2x$

13. $\lim_{x\to 0^-} e^{-2/x}$

14. $\lim_{x\to \infty} e^{-2/x}$

15. $\lim_{x\to \infty} \cot^{-1} x$

16. $\lim_{x\to \infty} \sec^{-1} x$

17. $\lim_{x\to \infty} e^{2x-1}$

18. $\lim_{x\to 0} e^{1/x^2}$

19. $\lim_{x\to \infty} \sin 2x$

20. $\lim_{x\to \infty} (e^{-3x} \cos 2x)$

21. $\lim_{x\to \infty} \frac{\ln(2+e^{3x})}{\ln(1+e^x)}$

22. $\lim_{x\to \infty} \sin(\tan^{-1} x)$

23. $\lim_{x\to \pi/2} e^{-\tan x}$

24. $\lim_{x\to 0^+} \tan^{-1}(\ln x)$

In exercises 25–34, determine all horizontal and vertical asymptotes. For each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$ on either side of the asymptote.

25. $f(x) = \frac{x}{\sqrt{4+x^2}}$

26. $f(x) = \frac{x}{\sqrt{4-x^2}}$

27. $f(x) = \frac{x}{4-x^2}$

28. $f(x) = \frac{x^2}{4-x^2}$

29. $f(x) = \frac{3x^2+1}{x^2-2x-3}$

30. $f(x) = \frac{1-x}{x^2+x-2}$

31. $f(x) = \ln(1-\cos x)$

32. $f(x) = \frac{\ln(x+2)}{\ln(x^2+3x+3)}$

33. $f(x) = 4\tan^{-1} x - 1$

34. $f(x) = 3e^{-1/x}$

In exercises 35–38, determine all vertical and slant asymptotes.

35. $y = \frac{x^3}{4-x^2}$

36. $y = \frac{x^2+1}{x-2}$

37. $y = \frac{x^3}{x^2+x-4}$

38. $y = \frac{x^4}{x^3+2}$

In exercises 39–48, use graphical and numerical evidence to conjecture a value for the indicated limit.

39. $\lim_{x\to \infty} \frac{x^2}{2^x}$

40. $\lim_{x\to \infty} \frac{2^x}{x^2}$

41. $\lim_{x\to \infty} \frac{x^2-4x+7}{2x^2+x\cos x}$

42. $\lim_{x\to -\infty} \frac{2x^3+7x^2+1}{x^3-x\sin x}$

43. $\lim_{x\to \infty} \frac{x^3+4x+5}{e^{x/2}}$

44. $\lim_{x\to \infty} (e^{x/3}-x^4)$

45. $\lim_{x\to -1} \frac{x-\cos(\pi x)}{x+1}$

46. $\lim_{x\to 0} \frac{e^x-1}{x}$

47. $\lim_{x\to 0^+} \frac{x}{\cos x - 1}$

48. $\lim_{x\to 0} \frac{\ln x^2}{x^2}$

In exercises 49–52, use graphical and numerical evidence to conjecture the value of the limit. Then, verify your conjecture by finding the limit exactly.

49. $\lim_{x\to\infty} (\sqrt{4x^2-2x+1}-2x)$ (Hint: Multiply and divide by the conjugate expression: $\sqrt{4x^2-2x+1}+2x$ and simplify.)

50. $\lim_{x\to\infty} (\sqrt{x^2+3}-x)$ (See the hint for exercise 49.)

51. $\lim_{x\to\infty} (\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3})$ (See the hint for exercise 49.)

52. $\lim_{x\to -\infty} \left(1+\frac{3}{x}\right)^{2x}$ $\left[\text{Hint}: e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.\right]$

53. Sketch the graph of $f(x)=e^{-x}\cos x$. Identify the horizontal asymptote. Is this asymptote approached in both directions (as $x\to\infty$ and $x\to-\infty$)? How many times does the graph cross the horizontal asymptote?

54. Explain why it is reasonable that $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(1/x)$ and $\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f(1/x)$.

55. In the exercises of section 1.2, we found that $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^-}(1+x)^{1/x}$. (It turns out that both limits equal the irrational number e.) Use this result and exercise 54 to argue that $\lim_{x\to\infty}(1+1/x)^x = \lim_{x\to-\infty}(1+1/x)^x$.

56. One of the reasons for saying that infinite limits do not exist is that we would otherwise invalidate Theorem 3.1 in section 1.3. Find examples of functions with infinite limits such that parts (ii) and (iv) of Theorem 3.1 do not hold.

57. Suppose that the length of a small animal t days after birth is $h(t) = \frac{300}{1+9(0.8)^t}$ mm. What is the length of the animal at birth? What is the eventual length of the animal (i.e., the length as $t\to\infty$)?

58. Suppose that the length of a small animal t days after birth is $h(t) = \frac{100}{2+3(0.4)^t}$ mm. What is the length of the animal at birth? What is the eventual length of the animal (i.e., the length as $t\to\infty$)?

59. Suppose that the size of the pupil of a certain animal is given by $f(x)$ (mm), where x is the intensity of the light on the pupil. If $f(x) = \frac{80x^{-0.3}+60}{2x^{-0.3}+5}$, find the size of the pupil with no light and the size of the pupil with an infinite amount of light.

60. Repeat exercise 59 with $f(x) = \frac{80x^{-0.3}+60}{8x^{-0.3}+15}$.

61. Modify the functions in exercises 59 and 60 to find a function f such that $\lim_{x\to 0^+} f(x) = 8$ and $\lim_{x\to\infty} f(x) = 2$.

62. After an injection, the concentration of a drug in a muscle varies according to a function of time $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Find the limit of $f(t)$ both as $t\to 0$ and $t\to\infty$, and interpret both limits in terms of the concentration of the drug.

63. Suppose an object with initial velocity $v_0 = 0$ ft/s and (constant) mass m slugs is accelerated by a constant force F pounds for t seconds. According to Newton's laws of motion, the object's speed will be $v_N = Ft/m$. According to Einstein's theory of relativity, the object's speed will be $v_E = Fct/\sqrt{m^2c^2+F^2t^2}$, where c is the speed of light. Compute $\lim_{t\to\infty} v_N$ and $\lim_{t\to\infty} v_E$.

64. According to Einstein's theory of relativity, the mass of an object traveling at speed v is given by $m = m_0/\sqrt{1-v^2/c^2}$, where c is the speed of light (about 9.8×10^8 ft/s). Compute $\lim_{v\to 0} m$ and explain why m_0 is called the "rest mass." Compute $\lim_{v\to c^-} m$ and discuss the implications. (What would happen if you were traveling in a spaceship approaching the speed of light?) How much does the mass of a 192-pound man ($m_0 = 6$) increase at the speed of 9000 ft/s (about 4 times the speed of sound)?

65. In example 5.12, the velocity of a skydiver t seconds after jumping is given by $v(t) = -\sqrt{\frac{32}{k}}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$. Find the limiting velocity with $k = 0.00064$ and $k = 0.00128$. By what factor does a skydiver have to change the value of k to cut the limiting velocity in half?

66. Graph the velocity function in exercise 65 with $k = 0.00016$ (representing a headfirst dive) and estimate how long it takes for the diver to reach a speed equal to 90% of the limiting velocity. Repeat with $k = 0.001$ (representing a spread-eagle position).

67. Ignoring air resistance, the maximum height reached by a rocket launched with initial velocity v_0 is $h = \frac{v_0^2 R}{19.6R - v_0^2}$ m/s, where R is the radius of the earth. In this exercise, we interpret this as a function of v_0. Explain why the domain of this function must be restricted to $v_0 \ge 0$. There is an additional restriction. Find the (positive) value v_e such that h is undefined. Sketch a possible graph of h with $0 \le v_0 < v_e$ and discuss the significance of the vertical asymptote at v_e. (Explain what would happen to the rocket if it is launched with initial velocity v_e.) Explain why v_e is called the **escape velocity.**

68. Suppose that $f(x)$ is a rational function $f(x) = \frac{p(x)}{q(x)}$ with the degree (largest exponent) of $p(x)$ less than the degree of $q(x)$. Determine the horizontal asymptote of $y = f(x)$.

69. Suppose that $f(x)$ is a rational function $f(x) = \frac{p(x)}{q(x)}$ with the degree of $p(x)$ greater than the degree of $q(x)$. Determine whether $y = f(x)$ has a horizontal asymptote.

70. Suppose that $f(x)$ is a rational function $f(x) = \frac{p(x)}{q(x)}$. If $y = f(x)$ has a horizontal asymptote $y = 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

71. Suppose that $f(x)$ is a rational function $f(x) = \frac{p(x)}{q(x)}$. If $y = f(x)$ has a slant asymptote $y = x + 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

72. Find a quadratic function $q(x)$ such that $f(x) = \frac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = 2$ and two vertical asymptotes $x = \pm 3$.

73. Find a quadratic function $q(x)$ such that $f(x) = \frac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = -\frac{1}{2}$ and exactly one vertical asymptote $x = 3$.

74. Find a function $g(x)$ such that $f(x) = \frac{x - 4}{g(x)}$ has two horizontal asymptotes $y = \pm 1$ and no vertical asymptotes.

In exercises 75–80, label the statement as true or false (not always true) for real numbers *a* and *b*.

75. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} [f(x) + g(x)] = a + b$.

76. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = \frac{a}{b}$.

77. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

78. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) + g(x)] = \infty$.

79. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = 0$.

80. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = 1$.

In exercises 81 and 82, determine all vertical and horizontal asymptotes.

81. $f(x) = \begin{cases} \frac{4x}{x-4} & \text{if } x < 0 \\ \frac{x^2}{x-2} & \text{if } 0 \le x < 4 \\ \frac{e^{-x}}{x+1} & \text{if } x \ge 4 \end{cases}$

82. $f(x) = \begin{cases} \frac{x+3}{x^2 - 4x} & \text{if } x < 0 \\ e^x + 1 & \text{if } 0 \le x < 2 \\ \frac{x^2 - 1}{x^2 - 7x + 10} & \text{if } x \ge 2 \end{cases}$

83. Explain why $\lim_{t\to\infty} (e^{-at} \sin t) = 0$ for any positive constant a. Although this is theoretically true, it is not necessarily useful in practice. The function $e^{-at} \sin t$ is a simple model for a spring-mass system, such as the suspension system on a car. Suppose t is measured in seconds and the car passengers cannot feel any vibrations less than 0.01 (inches). If suspension system A has the vibration function $e^{-t} \sin t$ and suspension system B has the vibration function $e^{-t/4} \sin t$, determine graphically how long it will take before the vibrations damp out, that is, $|f(t)| < 0.01$. Is the result $\lim_{t\to\infty} (e^{-at} \sin t) = 0$ much consolation to the owner of car B?

84. (a) State and prove a result analogous to Theorem 5.2 for $\lim_{x\to-\infty} p_n(x)$, for n odd.
(b) State and prove a result analogous to Theorem 5.2 for $\lim_{x\to-\infty} p_n(x)$, for n even.

85. It is very difficult to find simple statements in calculus that are always true; this is one reason that a careful development of the theory is so important. You may have heard the simple rule: to find the vertical asymptotes of $f(x) = \frac{g(x)}{h(x)}$, simply set the denominator equal to 0 [i.e., solve $h(x) = 0$]. Give an example where $h(a) = 0$ but there is *not* a vertical asymptote at $x = a$.

86. In exercise 85, you needed to find an example indicating that the following statement is not (necessarily) true: if $h(a) = 0$, then $f(x) = \frac{g(x)}{h(x)}$ has a vertical asymptote at $x = a$. This is not true, but perhaps its converse is true: if $f(x) = \frac{g(x)}{h(x)}$ has a vertical asymptote at $x = a$, then $h(a) = 0$. Is this statement true? What if g and h are polynomials?

In exercises 87–90, use numerical evidence to conjecture a decimal representation for the limit. Check your answer with your computer algebra system (CAS); if your answers disagree, which one is correct?

87. $\lim_{x\to 0^+} x^{1/\ln x}$

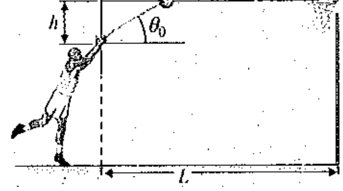
88. $\lim_{x\to 1^+} (\ln x)^{x^2-1}$

89. $\lim_{x\to\infty} x^{1/x}$

90. $\lim_{x\to-\infty} \frac{\ln x}{x^2}$

EXPLORATORY EXERCISES

1. Suppose you are shooting a basketball from a (horizontal) distance of L feet, releasing the ball from a location h feet below the basket. To get a perfect swish, it is necessary that the initial velocity v_0 and initial release angle θ_0 satisfy the equation



$v_0 = \sqrt{gL}/\sqrt{2\cos^2\theta_0(\tan\theta_0 - h/L)}$. For a free throw, take $L = 15$, $h = 2$ and $g = 32$ and graph v_0 as a function of θ_0.

What is the significance of the two vertical asymptotes? Explain in physical terms what type of shot corresponds to each vertical asymptote. Estimate the minimum value of v_0 (call it v_{min}). Explain why it is easier to shoot a ball with a small initial velocity. There is another advantage to this initial velocity. Assume that the basket is 2 ft in diameter and the ball is 1 ft in diameter. For a free throw, $L = 15$ ft is perfect. What is the maximum horizontal distance the ball could travel and still go in the basket (without bouncing off the backboard)? What is the minimum horizontal distance? Call these numbers L_{max} and L_{min}. Find the angle θ_1 corresponding to v_{min} and L_{min} and the angle θ_2 corresponding to v_{min} and L_{max}. The difference $|\theta_2 - \theta_1|$ is the angular margin of error. Brancazio has shown that the angular margin of error for v_{min} is larger than for any other initial velocity.

2. In applications, it is common to compute $\lim_{x\to\infty} f(x)$ to determine the "stability" of the function $f(x)$. Consider the function $f(x) = xe^{-x}$. As $x \to \infty$, the first factor in $f(x)$ goes to ∞, but the second factor goes to 0. What does the product do when one term is getting smaller and the other term is getting larger? It depends on which one is changing faster. What we want to know is which term "dominates." Use graphical and numerical evidence to conjecture the value of $\lim_{x\to\infty}(xe^{-x})$. Which term dominates? In the limit $\lim_{x\to\infty}(x^2e^{-x})$, which term dominates? Also, try $\lim_{x\to\infty}(x^5e^{-x})$. Based on your investigation, is it always true that exponentials dominate polynomials? Are you positive? Try to determine which type of function, polynomials or logarithms, dominates.

1.6 FORMAL DEFINITION OF THE LIMIT

We have now spent many pages discussing various aspects of the computation of limits. This may seem a bit odd, when you realize that we have never actually *defined* what a limit is. Oh, sure, we have given you an *idea* of what a limit is, but that's about all. Once again, we have said that

$$\lim_{x\to a} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a.

So far, we have been quite happy with this somewhat vague, although intuitive, description. In this section, however, we will make this more precise, and you will begin to see how **mathematical analysis** (that branch of mathematics of which the calculus is the most elementary study) works.

Studying more advanced mathematics without an understanding of the precise definition of limit is somewhat akin to studying brain surgery without bothering with all that background work in chemistry and biology. In medicine, it has only been through a careful examination of the microscopic world that a deeper understanding of our own macroscopic world has developed, and good surgeons need to understand what they are doing *and why* they are doing it. Likewise, in mathematical analysis, it is through an understanding of the microscopic behavior of functions (such as the precise definition of limit) that a deeper understanding of the mathematics will come about.

We begin with the careful examination of an elementary example. You should certainly believe that

$$\lim_{x\to 2}(3x + 4) = 10.$$

Suppose that you were asked to explain the meaning of this particular limit to a fellow student. You would probably repeat the intuitive explanation we have used so far: that as x gets closer and closer to 2, $(3x + 4)$ gets arbitrarily close to 10. But, exactly what do we mean by *close?* One answer is that if $\lim_{x\to 2}(3x + 4) = 10$, we should be able to make $(3x + 4)$ as close as we like to 10, just by making x sufficiently close to 2. But can we

HISTORICAL NOTES

Augustin Louis Cauchy (1789–1857) A French mathematician who developed the ε-δ definitions of limit and continuity. Cauchy was one of the most prolific mathematicians in history, making important contributions to number theory, linear algebra, differential equations, astronomy, optics and complex variables. A difficult man to get along with, a colleague wrote, "Cauchy is mad and there is nothing that can be done about him, although right now, he is the only one who knows how mathematics should be done."

What is the significance of the two vertical asymptotes? Explain in physical terms what type of shot corresponds to each vertical asymptote. Estimate the minimum value of v_0 (call it $v_{\min}$). Explain why it is easier to shoot a ball with a small initial velocity. There is another advantage to this initial velocity. Assume that the basket is 2 ft in diameter and the ball is 1 ft in diameter. For a free throw, $L = 15$ ft is perfect. What is the maximum horizontal distance the ball could travel and still go in the basket (without bouncing off the backboard)? What is the minimum horizontal distance? Call these numbers $L_{\max}$ and $L_{\min}$. Find the angle θ_1 corresponding to $v_{\min}$ and $L_{\min}$ and the angle θ_2 corresponding to $v_{\min}$ and $L_{\max}$. The difference $|\theta_2 - \theta_1|$ is the angular margin of error. Brancazio has shown that the angular margin of error for $v_{\min}$ is larger than for any other initial velocity.

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$$\lim_{x\to 2}(3x+4) = 10.$$

Suppose that you were asked to explain the meaning of this particular limit to a fellow student. You would probably repeat the intuitive explanation we have used so far: that as x gets closer and closer to 2, $(3x+4)$ gets arbitrarily close to 10. But, exactly what do we mean by *close?* One answer is that if $\lim_{x\to 2}(3x+4) = 10$, we should be able to make $(3x+4)$ as close as we like to 10, just by making x sufficiently close to 2. But can we

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actually do this? For instance, can we force $(3x + 4)$ to be within distance 1 of 10? To see what values of x will guarantee this, we write an inequality that says that $(3x + 4)$ is within 1 unit of 10:

$$|(3x + 4) - 10| < 1.$$

Eliminating the absolute values, we see that this is equivalent to

$$-1 < (3x + 4) - 10 < 1$$

or

$$-1 < 3x - 6 < 1.$$

Since we need to determine how close x must be to 2, we want to isolate $x - 2$, instead of x. So, dividing by 3, we get

$$-\frac{1}{3} < x - 2 < \frac{1}{3}$$

or

$$|x - 2| < \frac{1}{3}. \tag{6.1}$$

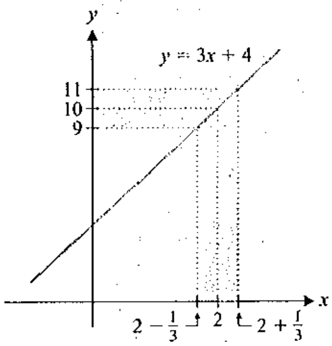

FIGURE 1.44
$2 - \frac{1}{3} < x < 2 + \frac{1}{3}$ guarantees that $|(3x + 4) - 10| < 1$

Reversing the steps that lead to inequality (6.1), we see that if x is within distance $\frac{1}{3}$ of 2, then $(3x + 4)$ will be within the specified distance (1) of 10. (See Figure 1.44 for a graphical interpretation of this.) So, does this convince you that you can make $(3x + 4)$ as close as you want to 10? Probably not, but if you used a smaller distance, perhaps you'd be more convinced.

EXAMPLE 6.1 **Exploring a Simple Limit**

Find the values of x for which $(3x + 4)$ is within distance $\frac{1}{100}$ of 10.

Solution We want

$$|(3x + 4) - 10| < \frac{1}{100}.$$

Eliminating the absolute values, we get

$$-\frac{1}{100} < (3x + 4) - 10 < \frac{1}{100}$$

or

$$-\frac{1}{100} < 3x - 6 < \frac{1}{100}.$$

Dividing by 3 yields

$$-\frac{1}{300} < x - 2 < \frac{1}{300},$$

which is equivalent to

$$|x - 2| < \frac{1}{300}.$$

So, based on example 6.1, are you now convinced that we can make $(3x + 4)$ as close as desired to 10? All we've been able to show is that we can make $(3x + 4)$ pretty close to 10. So, how close do we need to be able to make it? The answer is *arbitrarily close,* as close as anyone would ever demand. We can show that this is possible by repeating the arguments in example 6.1, this time for an unspecified distance, call it ε (*epsilon,* where $\varepsilon > 0$).

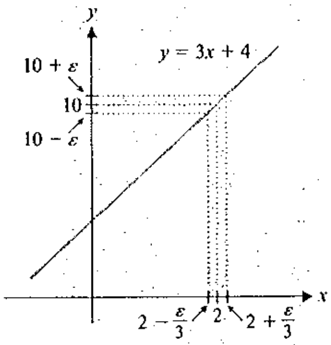

FIGURE 1.45
The range of x-values that keep $|(3x+4)-10| < \varepsilon$

EXAMPLE 6.2 Verifying a Limit

Show that we can make $(3x + 4)$ within any specified distance ε of 10 (no matter how small ε is), just by making x sufficiently close to 2.

Solution The objective is to determine the range of x-values that will guarantee that $(3x + 4)$ stays within ε of 10 (see Figure 1.45 for a sketch of this range). We have

$$|(3x+4)-10| < \varepsilon.$$

This is equivalent to $$-\varepsilon < (3x+4)-10 < \varepsilon$$

or $$-\varepsilon < 3x-6 < \varepsilon.$$

Dividing by 3, we get $$-\frac{\varepsilon}{3} < x-2 < \frac{\varepsilon}{3}$$

or $$|x-2| < \frac{\varepsilon}{3}.$$

Notice that each of the preceding steps is reversible, so that $|x-2| < \frac{\varepsilon}{3}$ also implies that $|(3x+4)-10| < \varepsilon$. This says that as long as x is within distance $\frac{\varepsilon}{3}$ of 2, $(3x + 4)$ will be within the required distance ε of 10. That is,

$$|(3x+4)-10| < \varepsilon \text{ whenever } |x-2| < \frac{\varepsilon}{3}.$$

Take a moment or two to recognize what we've done in example 6.2. By using an *unspecified* distance, ε, we have verified that we can indeed make $(3x + 4)$ as close to 10 as might be demanded (i.e., arbitrarily close; just name whatever $\varepsilon > 0$ you would like), simply by making x sufficiently close to 2. Further, we have explicitly spelled out what "sufficiently close to 2" means in the context of the present problem. Thus, no matter how close we are asked to make $(3x + 4)$ to 10, we can accomplish this simply by taking x to be in the specified interval.

Next, we examine this more precise notion of limit in the case of a function that is not defined at the point in question.

EXAMPLE 6.3 Proving That a Limit Is Correct

Prove that $\lim_{x\to 1} \frac{2x^2+2x-4}{x-1} = 6.$

Solution It is easy to use the usual rules of limits to establish this result. It is yet another matter to verify that this is correct using our new and more precise notion of limit. In this case, we want to know how close x must be to 1 to ensure that

$$f(x) = \frac{2x^2+2x-4}{x-1}$$

is within an unspecified distance $\varepsilon > 0$ of 6.

First, notice that f is undefined *at* $x = 1$. So, we seek a distance δ (*delta*, $\delta > 0$), such that if x is within distance δ of 1, but $x \neq 1$ (i.e., $0 < |x-1| < \delta$), then this guarantees that $|f(x) - 6| < \varepsilon$.

Notice that we have specified that $0 < |x - 1|$ to ensure that $x \neq 1$. Further, $|f(x) - 6| < \varepsilon$ is equivalent to

$$-\varepsilon < \frac{2x^2 + 2x - 4}{x - 1} - 6 < \varepsilon.$$

Finding a common denominator and subtracting in the middle term, we get

$$-\varepsilon < \frac{2x^2 + 2x - 4 - 6(x - 1)}{x - 1} < \varepsilon \quad \text{or} \quad -\varepsilon < \frac{2x^2 - 4x + 2}{x - 1} < \varepsilon.$$

Since the numerator factors, this is equivalent to

$$-\varepsilon < \frac{2(x - 1)^2}{x - 1} < \varepsilon.$$

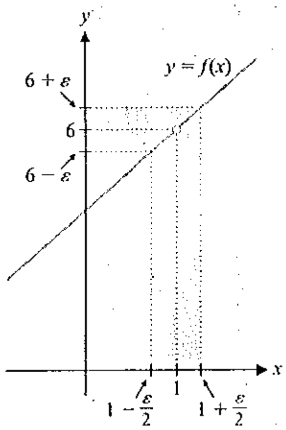

FIGURE 1.46
$0 < |x - 1| < \frac{\varepsilon}{2}$ guarantees that $6 - \varepsilon < \frac{2x^2 + 2x - 4}{x - 1} < 6 + \varepsilon$.

Since $x \neq 1$, we can cancel two of the factors of $(x - 1)$ to yield

$$-\varepsilon < 2(x - 1) < \varepsilon$$

or

$$-\frac{\varepsilon}{2} < x - 1 < \frac{\varepsilon}{2},$$

which is equivalent to $|x - 1| < \varepsilon/2$. So, taking $\delta = \varepsilon/2$ and working backward, we see that requiring x to satisfy

$$0 < |x - 1| < \delta = \frac{\varepsilon}{2}$$

will guarantee that

$$\left| \frac{2x^2 + 2x - 4}{x - 1} - 6 \right| < \varepsilon.$$

We illustrate this graphically in Figure 1.46.

What we have seen so far motivates us to make the following general definition, illustrated in Figure 1.47.

DEFINITION 6.1 (Precise Definition of Limit)

For a function f defined in some open interval containing a (but not necessarily *at a* itself), we say

$$\lim_{x \to a} f(x) = L,$$

if given any number $\varepsilon > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

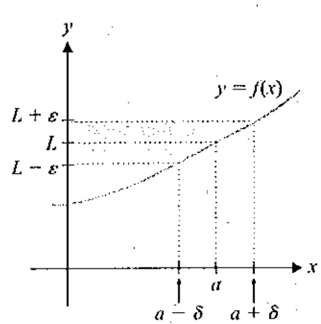

FIGURE 1.47
$a - \delta < x < a + \delta$ guarantees that $L - \varepsilon < f(x) < L + \varepsilon$.

Notice that example 6.2 amounts to an illustration of Definition 6.1 for $\lim_{x \to 2}(3x + 4)$. There, we found that $\delta = \varepsilon/3$ satisfies the definition.

REMARK 6.1

We want to emphasize that this formal definition of limit is not a new idea. Rather, it is a more precise mathematical statement of the same intuitive notion of limit that we have been using since the beginning of the chapter. Also, we must in all honesty point out that it is rather difficult to explicitly find δ as a function of ε, for all but a few simple examples. Despite this, learning how to work through the definition, even for a small number of problems, will shed considerable light on a deep concept.

TODAY IN MATHEMATICS

Paul Halmos (1916–) A Hungarian-born mathematician who earned a reputation as one of the best mathematical writers ever. For Halmos, calculus did not come easily, with understanding coming in a flash of inspiration only after a long period of hard work. "I remember standing at the blackboard in Room 213 of the mathematics building with Warren Ambrose and suddenly I understood epsilons. I understood what limits were, and all of that stuff that people had been drilling into me became clear. . . . I could prove the theorems. That afternoon I became a mathematician."

Example 6.4, although only slightly more complex than the last several problems, provides an unexpected challenge.

EXAMPLE 6.4 Using the Precise Definition of Limit

Use Definiton 6.1 to prove that $\lim_{x\to 2}(x^2 + 1) = 5$.

Solution If this limit is correct, then given any $\varepsilon > 0$, there must be a $\delta > 0$ for which $0 < |x - 2| < \delta$ guarantees that

$$|(x^2 + 1) - 5| < \varepsilon.$$

Notice that

$$\begin{aligned} |(x^2 + 1) - 5| &= |x^2 - 4| && \text{Factoring the difference} \\ &= |x + 2||x - 2|. && \text{of two squares.} \end{aligned} \qquad (6.2)$$

Our strategy is to isolate $|x - 2|$ and so, we'll need to do something with the term $|x + 2|$. Since we're interested only in what happens near $x = 2$, anyway, we will only consider x's within a distance of 1 from 2, that is, x's that lie in the interval $[1, 3]$ (so that $|x - 2| < 1$). Notice that this will be true if we require $\delta \leq 1$ and $|x - 2| < \delta$. In this case, we have

$$|x + 2| \leq 5, \quad \text{Since } x \in [1, 3].$$

and so, from (6.2),

$$\begin{aligned} |(x^2 + 1) - 5| &= |x + 2||x - 2| \\ &\leq 5|x - 2|. \end{aligned} \qquad (6.3)$$

Finally, if we require that

$$5|x - 2| < \varepsilon, \qquad (6.4)$$

then we will also have from (6.3) that

$$|(x^2 + 1) - 5| \leq 5|x - 2| < \varepsilon.$$

Of course, (6.4) is equivalent to

$$|x - 2| < \frac{\varepsilon}{5}.$$

So, in view of this, we now have two restrictions: that $|x - 2| < 1$ *and* that $|x - 2| < \frac{\varepsilon}{5}$. To ensure that both restrictions are met, we choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ (i.e., the **minimum** of 1 and $\frac{\varepsilon}{5}$). Working backward, we get that for this choice of δ,

$$0 < |x - 2| < \delta$$

will guarantee that

$$|(x^2 + 1) - 5| < \varepsilon,$$

as desired. We illustrate this in Figure 1.48. ■

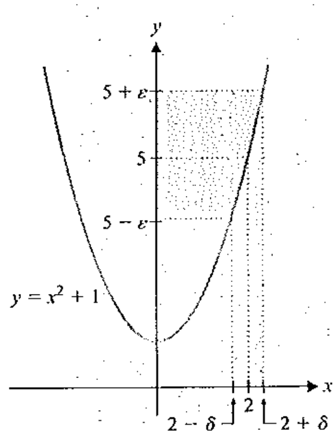

FIGURE 1.48
$0 < |x - 2| < \delta$ guarantees that $|(x^2 + 1) - 5| < \varepsilon$.

Exploring the Definition of Limit Graphically

As you can see from example 6.4, this business of finding δ's for a given ε is not easily accomplished. There, we found that even for the comparatively simple case of a quadratic polynomial, the job can be quite a challenge. Unfortunately, there is no procedure that will work for all problems. However, we can explore the definition graphically in the case of more complex functions. First, we reexamine example 6.4 graphically.

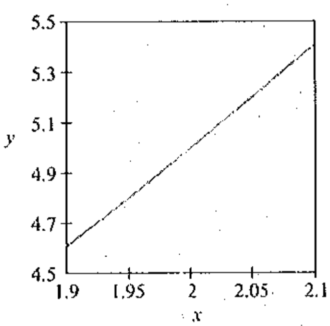

FIGURE 1.49
$y = x^2 + 1$

EXAMPLE 6.5 Exploring the Precise Definition of Limit Graphically

Explore the precise definition of limit graphically, for $\lim_{x\to 2}(x^2 + 1) = 5$.

Solution In example 6.4, we discovered that for $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$,

$$0 < |x - 2| < \delta \text{ implies that } |(x^2 + 1) - 5| < \varepsilon.$$

This says that (for $\varepsilon \le 5$) if we draw a graph of $y = x^2 + 1$ and restrict the x-values to lie in the interval $\left(2 - \frac{\varepsilon}{5}, 2 + \frac{\varepsilon}{5}\right)$, then the y-values will lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$. Take $\varepsilon = \frac{1}{2}$, for instance. If we draw the graph in the window defined by $2 - \frac{1}{10} \le x \le 2 + \frac{1}{10}$ and $4.5 \le y \le 5.5$, then the graph will not run off the top or bottom of the screen (see Figure 1.49). Of course, we can draw virtually the same picture for any given value of ε, since we have an explicit formula for finding δ given ε. For most limit problems, we are not so fortunate. ■

EXAMPLE 6.6 Exploring the Definition of Limit for a Trigonometric Function

Graphically find a $\delta > 0$ corresponding to (a) $\varepsilon = \frac{1}{2}$ and (b) $\varepsilon = 0.1$ for $\lim_{x\to 2} \sin\frac{\pi x}{2} = 0$.

Solution This limit seems plausible enough. After all, $\sin\frac{2\pi}{2} = 0$ and $f(x) = \sin x$ is a continuous function. However, the point is to verify this carefully. Given any $\varepsilon > 0$, we want to find a $\delta > 0$, for which

$$0 < |x - 2| < \delta \text{ guarantees that } \left|\sin\frac{\pi x}{2} - 0\right| < \varepsilon.$$

Note that since we have no algebra for simplifying $\sin\frac{\pi x}{2}$, we cannot accomplish this symbolically. Instead, we'll try to graphically find δ's corresponding to the specific ε's given. First, for $\varepsilon = \frac{1}{2}$, we would like to find a $\delta > 0$ for which if $0 < |x - 2| < \delta$, then

$$-\frac{1}{2} < \sin\frac{\pi x}{2} - 0 < \frac{1}{2}.$$

Drawing the graph of $y = \sin\frac{\pi x}{2}$ with $1 \le x \le 3$ and $-\frac{1}{2} \le y \le \frac{1}{2}$, we get Figure 1.50a.

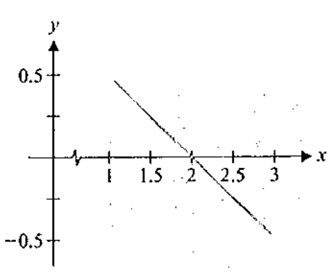

FIGURE 1.50a
$y = \sin\frac{\pi x}{2}$

If you trace along a calculator or computer graph, you will notice that the graph stays on the screen (i.e., the y-values stay in the interval $[-0.5, 0.5]$) for

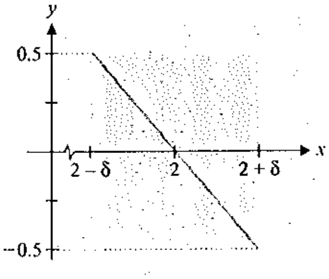

FIGURE 1.50b
$y = \sin\frac{\pi x}{2}$

$x \in [1.666667, 2.333333]$. Thus, we have determined experimentally that for $\varepsilon = \frac{1}{2}$,

$$\delta = 2.333333 - 2 = 2 - 1.666667 = 0.333333$$

will work. (Of course, any value of δ smaller than 0.333333 will also work.) To illustrate this, we redraw the last graph, but restrict x to lie in the interval [1.67, 2.33] (see Figure 1.50b). In this case, the graph stays in the window over the entire range of displayed x-values. Taking $\varepsilon = 0.1$, we look for an interval of x-values that will guarantee that $\sin\frac{\pi x}{2}$ stays between -0.1 and 0.1. We redraw the graph from Figure 1.50a, with the y-range restricted to the interval $[-0.1, 0.1]$ (see Figure 1.51a). Again, tracing along the graph tells us that the y-values will stay in the desired range for $x \in [1.936508, 2.063492]$. Thus, we have experimentally determined that

$$\delta = 2.063492 - 2 = 2 - 1.936508 = 0.063492$$

will work here. We redraw the graph using the new range of x-values (see Figure 1.51b), since the graph remains in the window for all values of x in the indicated interval.

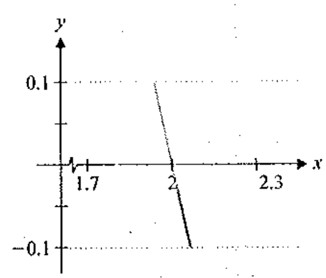

FIGURE 1.51a
$y = \sin\frac{\pi x}{2}$

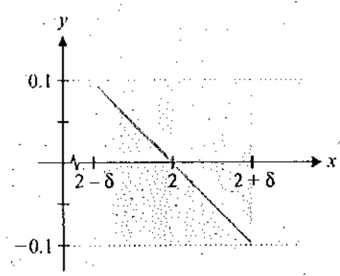

FIGURE 1.51b
$y = \sin\frac{\pi x}{2}$

It is important to recognize that we are not *proving* that the above limit is correct. To prove this requires us to symbolically find a δ for *every* $\varepsilon > 0$. The idea here is to use these graphical illustrations to become more familiar with the definition and with what δ and ε represent. ■

EXAMPLE 6.7 Exploring the Definition of Limit Where the Limit Does Not Exist

Determine whether or not $\lim_{x\to 0} \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} = 1$.

x	$\frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$
0.1	1.03711608
0.01	1.0037461
0.001	1.00037496
0.0001	1.0000375

Solution We first construct a table of function values. From the table alone, we might be tempted to conjecture that the limit is 1. However, we would be making a *huge* error, as we have not considered negative values of x or drawn a graph. This kind of carelessness is dangerous. Figure 1.52a (on the following page) shows the default graph drawn by our computer algebra system. In this graph, the function values do not quite look like they are approaching 1 as $x \to 0$ (at least as $x \to 0^-$). We now investigate the limit graphically for $\varepsilon = \frac{1}{2}$. We need to find a $\delta > 0$ for which $0 < |x| < \delta$ guarantees that

$$1 - \frac{1}{2} < \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} < 1 + \frac{1}{2}.$$

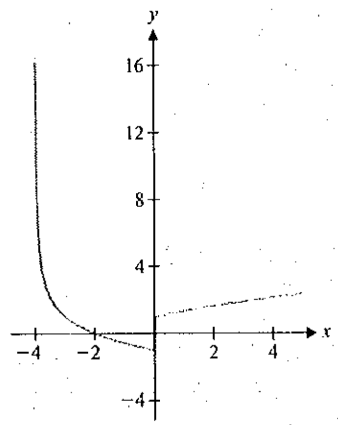

FIGURE 1.52a
$y = \dfrac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$

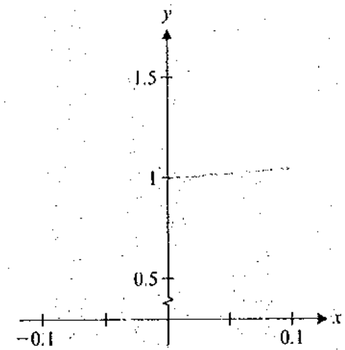

FIGURE 1.52b
$y = \dfrac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$

or

$$\frac{1}{2} < \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} < \frac{3}{2}.$$

We try $\delta = 0.1$ to see if this is sufficiently small. So, we set the x-range to the interval $[-0.1, 0.1]$ and the y-range to the interval $[0.5, 1.5]$ and redraw the graph in this window (see Figure 1.52b). Notice that no points are plotted in the window for any $x < 0$. According to the definition, the y-values must lie in the interval $(0.5, 1.5)$ for *all* x in the interval $(-\delta, \delta)$. Further, you can see that $\delta = 0.1$ clearly does not work since $x = -0.05$ lies in the interval $(-\delta, \delta)$, but $f(-0.05) \approx -0.981$ is not in the interval $(0.5, 1.5)$. You should convince yourself that no matter how small you make δ, there is an x in the interval $(-\delta, \delta)$ such that $f(x) \notin (0.5, 1.5)$. (In fact, notice that for all x's in the interval $(-1, 0)$, $f(x) < 0$.) That is, there is no choice of δ that makes the defining inequality true for $\varepsilon = \frac{1}{2}$. Thus, the conjectured limit of 1 is incorrect.

You should note here that, while we've only shown that the limit is not 1, it's somewhat more complicated to show that the limit does not exist. ■

○ Limits Involving Infinity

Recall that we had observed that $\lim\limits_{x\to 0} \frac{1}{x^2}$ does not exist, but to be more descriptive, we had written

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

By this statement, we mean that the function increases without bound as $x \to 0$. Just as with our initial intuitive notion of $\lim\limits_{x\to a} f(x) = L$, this description is imprecise and needs to be more carefully defined. When we say that $\frac{1}{x^2}$ increases without bound as $x \to 0$, we mean that we can make $\frac{1}{x^2}$ as large as we like, simply by making x sufficiently close to 0. So, given any large positive number, M, we must be able to make $\frac{1}{x^2} > M$, for x sufficiently close to 0. We measure closeness here the same way as we did before and arrive at the following definition.

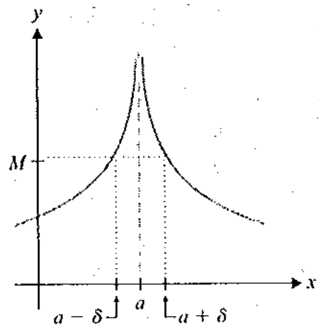

FIGURE 1.53
$\lim_{x \to a} f(x) = \infty$

DEFINITION 6.2

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = \infty,$$

if given any number $M > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) > M$. (See Figure 1.53 for a graphical interpretation of this.)

Similarly, we had said that if f decreases without bound as $x \to a$, then $\lim_{x \to a} f(x) = -\infty$. Think of how you would make this more precise and then consider the following definition.

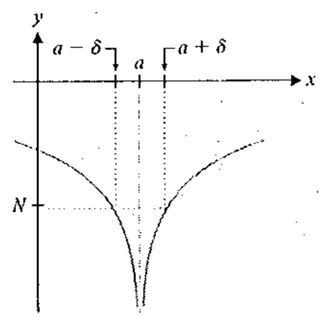

FIGURE 1.54
$\lim_{x \to a} f(x) = -\infty$

DEFINITION 6.3

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = -\infty,$$

if given any number $N < 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) < N$. (See Figure 1.54 for a graphical interpretation of this.)

It's easy to keep these definitions straight if you think of their meaning. Don't simply memorize them.

EXAMPLE 6.8 Using the Definition of Limit Where the Limit Is Infinite

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given any (large) number $M > 0$, we need to find a distance $\delta > 0$ such that if x is within δ of 0 (but not equal to 0) then

$$\frac{1}{x^2} > M. \tag{6.5}$$

Since both M and x^2 are positive, (6.5) is equivalent to

$$x^2 < \frac{1}{M}.$$

Taking the square root of both sides and recalling that $\sqrt{x^2} = |x|$, we get

$$|x| < \sqrt{\frac{1}{M}}.$$

So, for any $M > 0$, if we take $\delta = \sqrt{\frac{1}{M}}$ and work backward, we have that $0 < |x - 0| < \delta$ guarantees that

$$\frac{1}{x^2} > M,$$

as desired. Note that this says, for instance, that for $M = 100$, $\frac{1}{x^2} > 100$, whenever $0 < |x| < \sqrt{\frac{1}{100}} = \frac{1}{10}$. (Verify that this works, as an exercise.) ∎

There are two remaining limits that we have yet to place on a careful footing. Before reading on, try to figure out for yourself what appropriate definitions would look like.

If we write $\lim_{x\to\infty} f(x) = L$, we mean that as x increases without bound, $f(x)$ gets closer and closer to L. That is, we can make $f(x)$ as close to L as we like, by choosing x sufficiently large. More precisely, we have the following definition.

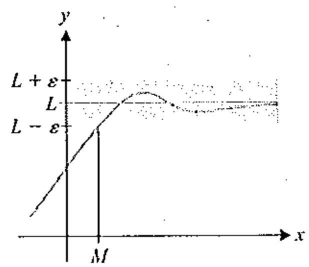

FIGURE 1.55
$\lim_{x\to\infty} f(x) = L$

DEFINITION 6.4

For a function f defined on an interval (a, ∞), for some $a > 0$, we say

$$\lim_{x\to\infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $M > 0$ such that $x > M$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 1.55 for a graphical interpretation of this.)

Similarly, we have said that $\lim_{x\to-\infty} f(x) = L$ means that as x decreases without bound, $f(x)$ gets closer and closer to L. So, we should be able to make $f(x)$ as close to L as desired, just by making x sufficiently large in absolute value and negative. We have the following definition.

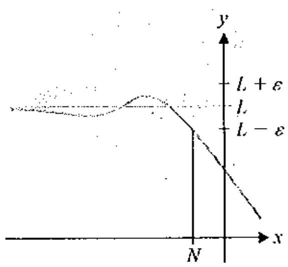

FIGURE 1.56
$\lim_{x\to-\infty} f(x) = L$

DEFINITION 6.5

For a function f defined on an interval $(-\infty, a)$, for some $a < 0$, we say

$$\lim_{x\to-\infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $N < 0$ such that $x < N$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 1.56 for a graphical interpretation of this.)

We use Definitions 6.4 and 6.5 essentially the same as we do Definitions 6.1–6.3, as we see in example 6.9.

EXAMPLE 6.9 Using the Definition of Limit Where x Is Becoming Infinite

Prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Solution Here, we must show that given any $\varepsilon > 0$, we can make $\frac{1}{x}$ within ε of 0, simply by making x sufficiently large in absolute value and negative. So, we need to determine those x's for which

$$\left|\frac{1}{x} - 0\right| < \varepsilon$$

or

$$\left|\frac{1}{x}\right| < \varepsilon. \tag{6.6}$$

REMARK 6.2

You should take care to note the commonality among the definitions of the five limits we have given. All five deal with a precise description of what it means to be "close." It is of considerable benefit to work through these definitions until you can provide your own words for each. Don't just memorize the formal definitions as stated here. Rather, work toward understanding what they mean and come to appreciate the exacting language mathematicians use.

Since $x < 0$, $|x| = -x$, and so (6.6) becomes

$$\frac{1}{-x} < \varepsilon.$$

Dividing both sides by ε and multiplying by x (remember that $x < 0$ and $\varepsilon > 0$, so that this will change the direction of the inequality), we get

$$-\frac{1}{\varepsilon} > x.$$

So, if we take $N = -\frac{1}{\varepsilon}$ and work backward, we have satisfied the definition and thereby proved that the limit is correct. ■

We don't use the limit definitions to prove each and every limit that comes along. Actually, we use them to prove only a few basic limits and to prove the limit theorems that we've been using for some time without proof. Further use of these theorems then provides solid justification of new limits. As an illustration, we now prove the rule for a limit of a sum.

THEOREM 6.1

Suppose that for a real number a, $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Then,

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2.$$

PROOF

Since $\lim_{x\to a} f(x) = L_1$, we know that given any number $\varepsilon_1 > 0$, there is a number $\delta_1 > 0$ for which

$$0 < |x - a| < \delta_1 \text{ guarantees that } |f(x) - L_1| < \varepsilon_1. \tag{6.7}$$

Likewise, since $\lim_{x\to a} g(x) = L_2$, we know that given any number $\varepsilon_2 > 0$, there is a number $\delta_2 > 0$ for which

$$0 < |x - a| < \delta_2 \text{ guarantees that } |g(x) - L_2| < \varepsilon_2. \tag{6.8}$$

Now, in order to get

$$\lim_{x\to a}[f(x) + g(x)] = (L_1 + L_2),$$

we must show that, given any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \text{ guarantees that } |[f(x) + g(x)] - (L_1 + L_2)| < \varepsilon.$$

Notice that

$$\begin{aligned} |[f(x) + g(x)] - (L_1 + L_2)| &= |[f(x) - L_1] + [g(x) - L_2]| \\ &\le |f(x) - L_1| + |g(x) - L_2|, \end{aligned} \tag{6.9}$$

by the triangle inequality. Of course, both terms on the right-hand side of (6.9) can be made arbitrarily small, from (6.7) and (6.8). In particular, if we take $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, then as long as

$$0 < |x - a| < \delta_1 \quad \textit{and} \quad 0 < |x - a| < \delta_2,$$

we get from (6.7), (6.8) and (6.9) that

$$|[f(x)+g(x)]-(L_1+L_2)| \le |f(x)-L_1|+|g(x)-L_2| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

as desired. Of course, this will happen if we take

$$0<|x-a|<\delta=\min\{\delta_1,\delta_2\}. \blacksquare$$

The other rules for limits are proven similarly, using the definition of limit. We show these in Appendix A.

EXERCISES 1.6

WRITING EXERCISES

1. In his 1726 masterpiece *Mathematical Principles of Natural Philosophy*, which introduces many of the fundamentals of calculus, Sir Isaac Newton described the important limit $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ (which we study at length in Chapter 2) as "the limit to which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor ever reach until the quantities vanish." If you ever get weary of all the notation that we use in calculus, think of what it would look like in words! Critique Newton's definition of limit, addressing the following questions in the process. What restrictions do the phrases "never go beyond" and "never reach" put on the limit process? Give an example of a simple limit, not necessarily of the form $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, that violates these restrictions. Give your own (English language) description of the limit, avoiding restrictions such as Newton's. Why do mathematicians consider the ε–δ definition simple and elegant?
2. You have computed numerous limits before seeing the definition of limit. Explain how this definition changes and/or improves your understanding of the limit process.
3. Each word in the ε–δ definition is carefully chosen and precisely placed. Describe what is wrong with each of the following slightly incorrect "definitions" (use examples!):
   (a) There exists $\varepsilon>0$ such that there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.
   (b) For all $\varepsilon>0$ and for all $\delta>0$, if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.
   (c) For all $\delta>0$ there exists $\varepsilon>0$ such that $0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$.
4. In order for the limit to exist, given every $\varepsilon>0$, we must be able to find a $\delta>0$ such that the if/then inequalities are true. To prove that the limit does not exist, we must find a particular $\varepsilon>0$ such that the if/then inequalities are not true for any choice of $\delta>0$. To understand the logic behind the swapping of the "for every" and "there exists" roles, draw an analogy with the following situation. Suppose the statement, "Everybody loves somebody" is true. If you wanted to verify the statement, why would you have to talk to every person on earth? But, suppose that the statement is not true. What would you have to do to disprove it?

In exercises 1–8, numerically and graphically determine a δ corresponding to (a) $\varepsilon=0.1$ and (b) $\varepsilon=0.05$. Graph the function in the $\varepsilon-\delta$ window [x-range is $(a-\delta, a+\delta)$ and y-range is $(L-\varepsilon, L+\varepsilon)$] to verify that your choice works.

1. $\lim_{x\to 0}(x^2+1)=1$
2. $\lim_{x\to 2}(x^2+1)=5$
3. $\lim_{x\to 0}\cos x=1$
4. $\lim_{x\to \pi/2}\cos x=0$
5. $\lim_{x\to 1}\sqrt{x+3}=2$
6. $\lim_{x\to -2}\sqrt{x+3}=1$
7. $\lim_{x\to 1}\frac{x+2}{x^2}=3$
8. $\lim_{x\to 2}\frac{x+2}{x^2}=1$

In exercises 9–20, symbolically find δ in terms of ε.

9. $\lim_{x\to 0}3x=0$
10. $\lim_{x\to 1}3x=3$
11. $\lim_{x\to 2}(3x+2)=8$
12. $\lim_{x\to 1}(3x+2)=5$
13. $\lim_{x\to 1}(3-4x)=-1$
14. $\lim_{x\to -1}(3-4x)=7$
15. $\lim_{x\to 1}\frac{x^2+x-2}{x-1}=3$
16. $\lim_{x\to -1}\frac{x^2-1}{x+1}=-2$
17. $\lim_{x\to 1}(x^2-1)=0$
18. $\lim_{x\to 1}(x^2-x+1)=1$
19. $\lim_{x\to 2}(x^2-1)=3$
20. $\lim_{x\to 0}(x^3+1)=1$
21. Determine a formula for δ in terms of ε for $\lim_{x\to a}(mx+b)$. (Hint: Use exercises 9–14.) Does the formula depend on the value of a? Try to explain this answer graphically.
22. Based on exercises 17 and 19, does the value of δ depend on the value of a for $\lim_{x\to a}(x^2+b)$? Try to explain this graphically.

23. Modify the $\varepsilon - \delta$ definition to define the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

24. Symbolically find the largest δ corresponding to $\varepsilon = 0.1$ in the definition of $\lim_{x\to 1^-} 1/x = 1$. Symbolically find the largest δ corresponding to $\varepsilon = 0.1$ in the definition of $\lim_{x\to 1^+} 1/x = 1$. Which δ could be used in the definition of $\lim_{x\to 1} 1/x = 1$? Briefly explain. Then prove that $\lim_{x\to 1} 1/x = 1$.

In exercises 25–30, find a δ corresponding to $M = 100$ or $N = -100$ (as appropriate) for each limit.

25. $\lim_{x\to 1^+} \frac{2}{x-1} = \infty$

26. $\lim_{x\to 1^-} \frac{2}{x-1} = -\infty$

27. $\lim_{x\to 0^+} \cot x = \infty$

28. $\lim_{x\to \pi^-} \cot x = -\infty$

29. $\lim_{x\to 2^-} \frac{2}{\sqrt{4-x^2}} = \infty$

30. $\lim_{x\to 0^+} \ln x = -\infty$

In exercises 31–36, find an M or N corresponding to $\varepsilon = 0.1$ for each limit at infinity.

31. $\lim_{x\to\infty} \frac{x^2-2}{x^2+x+1} = 1$

32. $\lim_{x\to\infty} \frac{x-2}{x^2+x+1} = 0$

33. $\lim_{x\to-\infty} \frac{x^2+3}{4x^2-4} = 0.25$

34. $\lim_{x\to-\infty} \frac{3x^2-2}{x^2+1} = 3$

35. $\lim_{x\to\infty} e^{-2x} = 0$

36. $\lim_{x\to\infty} \frac{e^x+x}{e^x-x^2} = 1$

In exercises 37–46, prove that the limit is correct using the appropriate definition (assume that k is an integer).

37. $\lim_{x\to\infty} \frac{2}{x^3} = 0$

38. $\lim_{x\to-\infty} \frac{3}{x^3} = 0$

39. $\lim_{x\to\infty} \frac{1}{x^k} = 0$, for $k > 0$

40. $\lim_{x\to-\infty} \frac{1}{x^{2k}} = 0$, for $k > 0$

41. $\lim_{x\to\infty} \left(\frac{1}{x^2+2} - 3\right) = -3$

42. $\lim_{x\to\infty} \frac{1}{(x-7)^2} = 0$

43. $\lim_{x\to-3} \frac{-2}{(x+3)^4} = -\infty$

44. $\lim_{x\to 7} \frac{3}{(x-7)^2} = \infty$

45. $\lim_{x\to 5} \frac{4}{(x-5)^2} = \infty$

46. $\lim_{x\to-4} \frac{-6}{(x+4)^6} = -\infty$

In exercises 47–50, identify a specific $\varepsilon > 0$ for which no $\delta > 0$ exists to satisfy the definition to limit.

47. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2+3 & \text{if } x > 1 \end{cases}$, $\lim_{x\to 1} f(x) \neq 2$

48. $f(x) = \begin{cases} x^2-1 & \text{if } x < 0 \\ -x-2 & \text{if } x > 0 \end{cases}$, $\lim_{x\to 0} f(x) \neq -2$

49. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 5-x^2 & \text{if } x > 1 \end{cases}$, $\lim_{x\to 1} f(x) \neq 2$

50. $f(x) = \begin{cases} x-1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$, $\lim_{x\to 2} f(x) \neq 1$

51. A metal washer of (outer) radius r inches weighs $2r^2$ ounces. A company manufactures 2-inch washers for different customers who have different error tolerances. If the customer demands a washer of weight $8 \pm \varepsilon$ ounces, what is the error tolerance for the radius? That is, find δ such that a radius of r within the interval $(2-\delta, 2+\delta)$ guarantees a weight within $(8-\varepsilon, 8+\varepsilon)$.

52. A fiberglass company ships its glass as spherical marbles. If the volume of each marble must be within ε of $\pi/6$, how close does the radius need to be to $1/2$?

53. Prove Theorem 3.1 (i).

54. Prove Theorem 3.1 (ii).

55. Prove the Squeeze Theorem, as stated in Theorem 3.5.

56. Given that $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, prove that $\lim_{x\to a} f(x) = L$.

57. Prove: if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a}[f(x) - L] = 0$.

58. Prove: if $\lim_{x\to a}[f(x) - L] = 0$, then $\lim_{x\to a} f(x) = L$.

59. In this exercise, we explore the definition of $\lim_{x\to 2} x^2 = 4$ with $\varepsilon = 0.1$. Show that $x^2 - 4 < 0.1$ if $2 < x < \sqrt{4.1}$. This indicates that $\delta_1 = 0.02484$ works for $x > 2$. Show that $x^2 - 4 > -0.1$ if $\sqrt{3.9} < x < 2$. This indicates that $\delta_2 = 0.02515$ works for $x < 2$. For the limit definition, is $\delta = \delta_1$ or $\delta = \delta_2$ the correct choice? Briefly explain.

60. Generalize exercise 59 to find a δ of the form $\sqrt{4+\varepsilon}$ or $\sqrt{4-\varepsilon}$ corresponding to any $\varepsilon > 0$.

EXPLORATORY EXERCISES

1. We hope that working through this section has provided you with extra insight into the limit process. However, we have not yet solved any problems we could not already solve in previous sections. We do so now, while investigating an unusual function. Recall that rational numbers can be written as fractions p/q, where p and q are integers. We will assume that p/q has been simplified by dividing out common factors (e.g., $1/2$ and not $2/4$). Define $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = \frac{p}{q} \text{ is rational} \end{cases}$. We will try to show that $\lim_{x\to 2/3} f(x)$ exists. Without graphics,

we need a good definition to answer this question. We know that $f(2/3) = 1/3$, but recall that the limit is independent of the actual function value. We need to think about x's close to 2/3. If such an x is irrational, $f(x) = 0$. A simple hypothesis would then be $\lim_{x\to 2/3} f(x) = 0$. We'll try this out for $\varepsilon = 1/6$. We would like to guarantee that $|f(x)| < 1/6$ whenever $0 < |x - 2/3| < \delta$. Well, how many x's have a function value greater than 1/6? The only possible function values are 1/5, 1/4, 1/3, 1/2 and 1. The x's with function value 1/5 are 1/5, 2/5, 3/5, 4/5 and so on. The closest of these x's to 2/3 is 3/5. Find the closest x (not counting $x = 2/3$) to 2/3 with function value 1/4. Repeat for $f(x) = 1/3$, $f(x) = 1/2$ and $f(x) = 1$. Out of all these closest x's, how close is the absolute closest? Choose δ to be this number, and argue that if $0 < |x - 2/3| < \delta$, we are guaranteed that $|f(x)| < 1/6$. Argue that a similar process can find a δ for any ε.

2. State a definition for "$f(x)$ is continuous at $x = a$" using Definition 6.1. Use it to prove that the function in exploratory exercise 1 is continuous at every irrational number and discontinuous at every rational number.

1.7 LIMITS AND LOSS-OF-SIGNIFICANCE ERRORS

"Pay no attention to that man behind the curtain . . . ." (from *The Wizard of Oz*)

Things are not always what they appear to be. We spend much time learning to distinguish reality from mere appearances. Along the way, we develop a healthy level of skepticism. You may have already come to realize that mathematicians are a skeptical lot. This is of necessity, for you simply can't accept things at face value.

People tend to accept a computer's answer as a fact not subject to debate. However, when we use a computer (or calculator), we must always keep in mind that these devices perform most computations only approximately. Most of the time, this will cause us no difficulty whatsoever. Modern computational devices generally carry out calculations to a very high degree of accuracy. Occasionally, however, the results of round-off errors in a string of calculations are disastrous. In this section, we briefly investigate these errors and learn how to recognize and avoid some of them.

We first consider a relatively tame-looking example.

EXAMPLE 7.1 A Limit with Unusual Graphical and Numerical Behavior

Evaluate $\displaystyle\lim_{x\to\infty} \frac{(x^3+4)^2 - x^6}{x^3}$.

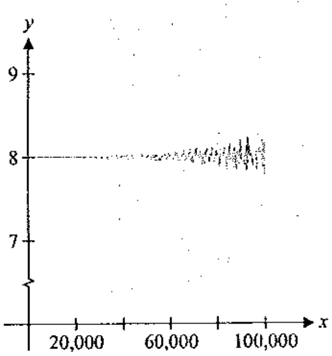

FIGURE 1.57a

$y = \dfrac{(x^3+4)^2 - x^6}{x^3}$

Solution At first glance, the numerator looks like $\infty - \infty$, which is indeterminate, while the denominator tends to ∞. Algebraically, the only reasonable step to take is to multiply out the first term in the numerator. Before we do that, let's draw a graph and compute some function values. (Different computers and different software will produce somewhat different results, but for large values of x, you should see results similar to those shown here.) In Figure 1.57a, the function appears nearly constant, until it begins oscillating around $x = 40{,}000$. Notice that the accompanying table of function values is inconsistent with Figure 1.57a.

The last two values in the table may have surprised you. Up until that point, the function values seemed to be settling down to 8.0 very nicely. So, what happened here and what is the correct value of the limit? Obviously, something unusual has occurred we need a good definition to answer this question. We know that $f(2/3) = 1/3$, but recall that the limit is independent of the actual function value. We need to think about x's close to 2/3. If such an x is irrational, $f(x) = 0$. A simple hypothesis would then be $\lim_{x \to 2/3} f(x) = 0$. We'll try this out for $\varepsilon = 1/6$. We would like to guarantee that $|f(x)| < 1/6$ whenever $0 < |x - 2/3| < \delta$. Well, how many x's have a function value greater than 1/6? The only possible function values are 1/5, 1/4, 1/3, 1/2 and 1. The x's with function value 1/5 are 1/5, 2/5, 3/5, 4/5 and so on. The closest of these x's to 2/3 is 3/5. Find the closest x (not counting $x = 2/3$) to 2/3 with function value 1/4. Repeat for $f(x) = 1/3$, $f(x) = 1/2$ and $f(x) = 1$. Out of all these closest x's, how close is the absolute closest? Choose δ to be this number, and argue that if $0 < |x - 2/3| < \delta$, we are guaranteed that $|f(x)| < 1/6$. Argue that a similar process can find a δ for any ε.

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We first consider a relatively tame-looking example.

EXAMPLE 7.1 A Limit with Unusual Graphical and Numerical Behavior

Evaluate $\lim_{x \to \infty} \frac{(x^3 + 4)^2 - x^6}{x^3}$.

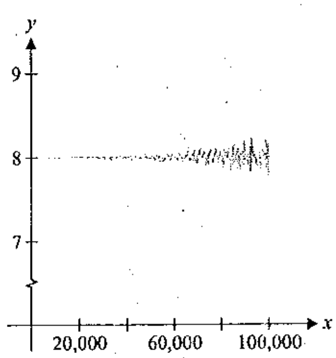

FIGURE 1.57a
$y = \frac{(x^3 + 4)^2 - x^6}{x^3}$

Solution At first glance, the numerator looks like $\infty - \infty$, which is indeterminate, while the denominator tends to ∞. Algebraically, the only reasonable step to take is to multiply out the first term in the numerator. Before we do that, let's draw a graph and compute some function values. (Different computers and different software will produce somewhat different results, but for large values of x, you should see results similar to those shown here.) In Figure 1.57a, the function appears nearly constant, until it begins oscillating around $x = 40{,}000$. Notice that the accompanying table of function values is inconsistent with Figure 1.57a.

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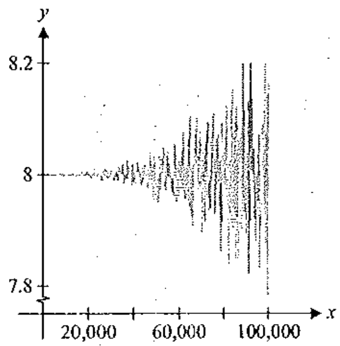


FIGURE 1.57b
$y = \dfrac{(x^3+4)^2 - x^6}{x^3}$

between $x = 1 \times 10^4$ and $x = 1 \times 10^5$. We should look carefully at function values in that interval. A more detailed table is shown below to the right.

Incorrect calculated values

x	$\dfrac{(x^3+4)^2 - x^6}{x^3}$
10	8.016
100	8.000016
1×10^3	8.0
1×10^4	8.0
1×10^5	0.0
1×10^6	0.0

x	$\dfrac{(x^3+4)^2 - x^6}{x^3}$
2×10^4	8.0
3×10^4	8.14815
4×10^4	7.8125
5×10^4	0

In Figure 1.57b, we have blown up the graph to enhance the oscillation observed between $x = 1 \times 10^4$ and $x = 1 \times 10^5$. The picture that is emerging is even more confusing. The deeper we look into this limit, the more erratically the function appears to behave. We use the word *appears* because all of the oscillatory behavior we are seeing is an illusion, created by the finite precision of the computer used to perform the calculations or draw the graph. ■

○ Computer Representation of Real Numbers

The reason for the unusual behavior seen in example 7.1 boils down to the way in which computers represent real numbers. Without getting into all of the intricacies of computer arithmetic, it suffices to think of computers and calculators as storing real numbers internally in scientific notation. For example, the number 1,234,567 would be stored as 1.234567×10^6. The number preceding the power of 10 is called the **mantissa** and the power is called the **exponent.** Thus, the mantissa here is 1.234567 and the exponent is 6.

All computing devices have finite memory and consequently have limitations on the size mantissa and exponent that they can store. (This is called **finite precision.**) Many calculators carry a 14-digit mantissa and a 3-digit exponent. On a 14-digit computer, this would suggest that the computer would retain only the first 14 digits in the decimal expansion of any given number.

EXAMPLE 7.2 Computer Representation of a Rational Number

Determine how $\frac{1}{3}$ is stored internally on a 10-digit computer and how $\frac{2}{3}$ is stored internally on a 14-digit computer.

Solution On a 10-digit computer, $\frac{1}{3}$ is stored internally as $\underbrace{3.333333333}_{10\text{ digits}} \times 10^{-1}$. On a 14-digit computer, $\frac{2}{3}$ is stored internally as $\underbrace{6.6666666666667}_{14\text{ digits}} \times 10^{-1}$. ■

For most purposes, such finite precision presents no problem. However, we do occasionally come across a disastrous error caused by finite precision. In example 7.3, we subtract two relatively close numbers and examine the resulting catastrophic error.

EXAMPLE 7.3 A Computer Subtraction of Two "Close" Numbers

Compare the exact value of

$$1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{18} - 1.\underbrace{0000000000000}_{13\text{ zeros}}1 \times 10^{18}$$

with the result obtained from a calculator or computer with a 14-digit mantissa.

Solution Notice that

$$1.\underbrace{0000000000000}_{13\text{ zeros}}4\times 10^{18} - 1.\underbrace{0000000000000}_{13\text{ zeros}}1\times 10^{18} = 0.\underbrace{0000000000000}_{13\text{ zeros}}3\times 10^{18}$$
$$= 30{,}000. \tag{7.1}$$

However, if this calculation is carried out on a computer or calculator with a 14-digit (or smaller) mantissa, both numbers on the left-hand side of (7.1) are stored by the computer as 1×10^{18} and hence, the difference is calculated as 0. Try this calculation for yourself now. ■

EXAMPLE 7.4 Another Subtraction of Two "Close" Numbers

Compare the exact value of

$$1.\underbrace{0000000000000}_{13\text{ zeros}}6 \times 10^{20} - 1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{20}$$

with the result obtained from a calculator or computer with a 14-digit mantissa.

Solution Notice that

$$1.\underbrace{0000000000000}_{13\text{ zeros}}6\times 10^{20} - 1.\underbrace{0000000000000}_{13\text{ zeros}}4\times 10^{20} = 0.\underbrace{0000000000000}_{13\text{ zeros}}2\times 10^{20}$$
$$= 2{,}000{,}000.$$

However, if this calculation is carried out on a calculator with a 14-digit mantissa, the first number is represented as $1.0000000000001 \times 10^{20}$, while the second number is represented by 1.0×10^{20}, due to the finite precision and rounding. The difference between the two values is then computed as $0.0000000000001 \times 10^{20}$ or 10,000,000, which is, again, a very serious error. ■

In examples 7.3 and 7.4, we witnessed a gross error caused by the subtraction of two numbers whose significant digits are very close to one another. This type of error is called a **loss-of-significant-digits error** or simply a **loss-of-significance error**. These are subtle, often disastrous computational errors. Returning now to example 7.1, we will see that it was this type of error that caused the unusual behavior noted.

EXAMPLE 7.5 A Loss-of-Significance Error

In example 7.1, we considered the function $f(x) = \dfrac{(x^3+4)^2 - x^6}{x^3}$. Follow the calculation of $f(5 \times 10^4)$ one step at a time, as a 14-digit computer would do it.

Solution We have

$$f(5\times 10^4) = \frac{[(5\times 10^4)^3+4]^2-(5\times 10^4)^6}{(5\times 10^4)^3}$$
$$= \frac{(1.25\times 10^{14}+4)^2-1.5625\times 10^{28}}{1.25\times 10^{14}}$$
$$= \frac{(125{,}000{,}000{,}000{,}000+4)^2-1.5625\times 10^{28}}{1.25\times 10^{14}}$$
$$= \frac{(1.25\times 10^{14})^2-1.5625\times 10^{28}}{1.25\times 10^{14}} = 0,$$

since 125,000,000,000,004 is rounded off to 125,000,000,000,000.

Note that the real culprit here was not the rounding of 125,000,000,000,004, but the fact that this was followed by a subtraction of a nearly equal value. Further, note that this is not a problem unique to the numerical computation of limits, but one that occurs in numerical computation, in general. ■

REMARK 7.1

If at all possible, avoid subtractions of nearly equal values. Sometimes, this can be accomplished by some algebraic manipulation of the function.

In the case of the function from example 7.5, we can avoid the subtraction and hence, the loss-of-significance error by rewriting the function as follows:

$$f(x) = \frac{(x^3+4)^2-x^6}{x^3}$$
$$= \frac{(x^6+8x^3+16)-x^6}{x^3}$$
$$= \frac{8x^3+16}{x^3},$$

where we have eliminated the subtraction. Using this new (and equivalent) expression for the function, we can compute a table of function values reliably. Notice, too, that if we redraw the graph in Figure 1.57a using the new expression (see Figure 1.58), we no longer see the oscillation present in Figures 1.57a and 1.57b.

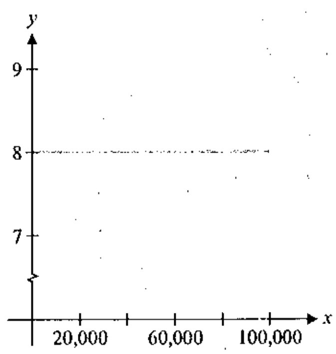


FIGURE 1.58
$y = \frac{8x^3+16}{x^3}$

x	$\frac{8x^3+16}{x^3}$
10	8.016
100	8.000016
1×10^3	8.000000016
1×10^4	8.00000000002
1×10^5	8.0
1×10^6	8.0
1×10^7	8.0

From the rewritten expression, we easily obtain

$$\lim_{x\to\infty} \frac{(x^3+4)^2-x^6}{x^3} = 8,$$

which is consistent with Figure 1.58 and the corrected table of function values.

In example 7.6, we examine a loss-of-significance error that occurs for x close to 0.

EXAMPLE 7.6 Loss-of-Significance Involving a Trigonometric Function

Evaluate $\lim\limits_{x\to 0} \frac{1-\cos x^2}{x^4}$.

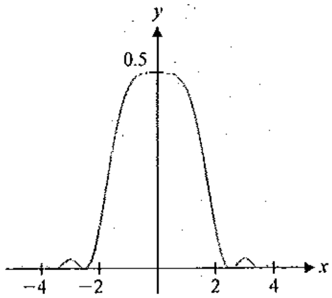


FIGURE 1.59
$y = \dfrac{1 - \cos x^2}{x^4}$

Solution As usual, we look at a graph (see Figure 1.59) and some function values.

x	$\dfrac{1 - \cos x^2}{x^4}$
0.1	0.499996
0.01	0.5
0.001	0.5
0.0001	0.0
0.00001	0.0

x	$\dfrac{1 - \cos x^2}{x^4}$
−0.1	0.499996
−0.01	0.5
−0.001	0.5
−0.0001	0.0
−0.00001	0.0

As in example 7.1, note that the function values seem to be approaching 0.5, but then suddenly take a jump down to 0.0. Once again, we are seeing a loss-of-significance error. In this particular case, this occurs because we are subtracting nearly equal values ($\cos x^2$ and 1). We can again avoid the error by eliminating the subtraction. Note that

$$\begin{aligned}\frac{1-\cos x^2}{x^4} &= \frac{1-\cos x^2}{x^4}\cdot\frac{1+\cos x^2}{1+\cos x^2} && \text{Multiply numerator and denominator by } (1+\cos x^2).\\ &= \frac{1-\cos^2(x^2)}{x^4\left(1+\cos x^2\right)} && 1-\cos^2(x^2)=\sin^2(x^2)\\ &= \frac{\sin^2(x^2)}{x^4\left(1+\cos x^2\right)}.\end{aligned}$$

x	$\dfrac{\sin^2(x^2)}{x^4(1 + \cos x^2)}$
±0.1	0.499996
±0.01	0.4999999996
±0.001	0.5
±0.0001	0.5
±0.00001	0.5

Since this last (equivalent) expression has no subtraction indicated, we should be able to use it to reliably generate values without the worry of loss-of-significance error. Using this to compute function values, we get the accompanying table.

Using the graph and the new table, we conjecture that

$$\lim_{x\to 0} \frac{1 - \cos x^2}{x^4} = \frac{1}{2}.$$

We offer one final example where a loss-of-significance error occurs, even though no subtraction is explicitly indicated.

EXAMPLE 7.7 A Loss-of-Significance Error Involving a Sum

Evaluate $\lim\limits_{x\to -\infty} x[(x^2+4)^{1/2} + x]$.

Solution Initially, you might think that since there is no subtraction (explicitly) indicated, there will be no loss-of-significance error. We first draw a graph (see Figure 1.60) and compute a table of values.

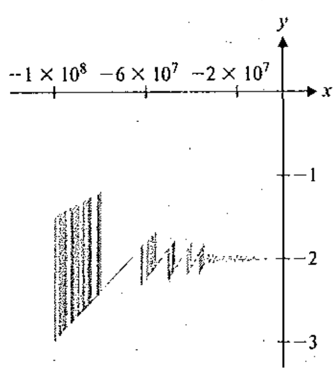


FIGURE 1.60
$y = x[(x^2+4)^{1/2} + x]$

x	$x\left[(x^2+4)^{1/2} + x\right]$
−100	−1.9998
-1×10^3	−1.999998
-1×10^4	−2.0
-1×10^5	−2.0
-1×10^6	−2.0
-1×10^7	0.0
-1×10^8	0.0

You should quickly notice the sudden jump in values in the table and the wild oscillation visible in the graph. Although a subtraction is not explicitly indicated, there is indeed a subtraction here, since we have $x < 0$ and $(x^2+4)^{1/2} > 0$. We can again remedy this with some algebraic manipulation, as follows.

$$x\left[(x^2+4)^{1/2}+x\right] = x\left[(x^2+4)^{1/2}+x\right]\frac{\left[(x^2+4)^{1/2}-x\right]}{\left[(x^2+4)^{1/2}-x\right]}$$

Multiply numerator and denominator by the conjugate.

$$= x\frac{\left[(x^2+4)-x^2\right]}{\left[(x^2+4)^{1/2}-x\right]}$$

Simplify the numerator.

$$= \frac{4x}{\left[(x^2+4)^{1/2}-x\right]}.$$

We use this last expression to generate a graph in the same window as that used for Figure 1.60 and to generate the accompanying table of values. In Figure 1.61, we can see none of the wild oscillation observed in Figure 1.60, and the graph appears to be a horizontal line.

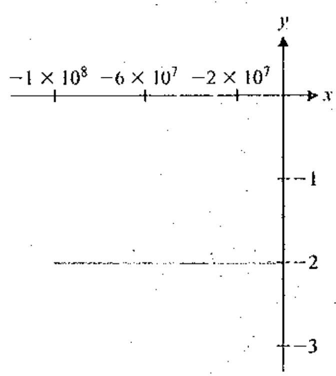


FIGURE 1.61
$y = \dfrac{4x}{[(x^2+4)^{1/2}-x]}$

x	$\dfrac{4x}{\left[(x^2+4)^{1/2}-x\right]}$
-100	-1.9998
-1×10^3	-1.999998
-1×10^4	-1.99999998
-1×10^5	-1.9999999998
-1×10^6	-2.0
-1×10^7	-2.0
-1×10^8	-2.0

Further, the values displayed in the table no longer show the sudden jump indicative of a loss-of-significance error. We can now confidently conjecture that

$$\lim_{x\to-\infty} x[(x^2+4)^{1/2}+x] = -2. \blacksquare$$

BEYOND FORMULAS

In examples 7.5–7.7, we demonstrated calculations that suffered from catastrophic loss-of-significance errors. In each case, we showed how we could rewrite the expression to avoid this error. We have by no means exhibited a general procedure for recognizing and repairing such errors. Rather, we hope that by seeing a few of these subtle errors, and by seeing how to fix even a limited number of them, you will become a more skeptical and intelligent user of technology.

EXERCISES 1.7

WRITING EXERCISES

1. It is probably clear that caution is important in using technology. Equally important is redundancy. This property is sometimes thought to be a negative (wasteful, unnecessary), but its positive role is one of the lessons of this section. By redundancy, we mean investigating a problem using graphical, numerical and symbolic tools. Why is it important to use multiple methods? Answer this from a practical perspective (think of the problems in this section) and a theoretical perspective (if you have learned multiple techniques, do you understand the mathematics better?).

2. The drawback of caution and redundancy is that they take extra time. In computing limits, when should you stop and take extra time to make sure an answer is correct and when is it safe to go on to the next problem? Should you always look at a graph? compute function values? do symbolic work? an ε–δ proof? Prioritize the techniques in this chapter.

3. The limit $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ will be very important in Chapter 2. For a specific function and specific a, we could compute a table of values of the fraction for smaller and smaller values of h. Why should we be wary of loss-of-significance errors?

4. Notice that we rationalized the numerator in example 7.7. The old rule of rationalizing the denominator is another example of rewriting an expression to try to minimize computational errors. Before computers, square roots were very difficult to compute. To see one reason why you might want the square root in the numerator, suppose that you can get only one decimal place of accuracy, so that $\sqrt{3}\approx 1.7$. Compare $\frac{6}{1.7}$ to $\frac{6}{\sqrt{3}}$ and then compare $2(1.7)$ to $\frac{6}{\sqrt{3}}$. Which of the approximations could you do in your head?

In exercises 1–12, (a) use graphics and numerics to conjecture a value of the limit. (b) Find a computer or calculator graph showing a loss-of-significance error. (c) Rewrite the function to avoid the loss-of-significance error.

1. $\lim\limits_{x\to\infty} x\left(\sqrt{4x^2+1}-2x\right)$

2. $\lim\limits_{x\to-\infty} x\left(\sqrt{4x^2+1}+2x\right)$

3. $\lim\limits_{x\to\infty} \sqrt{x}\left(\sqrt{x+4}-\sqrt{x+2}\right)$

4. $\lim\limits_{x\to\infty} x^2\left(\sqrt{x^4+8}-x^2\right)$

5. $\lim\limits_{x\to\infty} x\left(\sqrt{x^2+4}-\sqrt{x^2+2}\right)$

6. $\lim\limits_{x\to\infty} x\left(\sqrt{x^3+8}-x^{3/2}\right)$

7. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{12x^2}$

8. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

9. $\lim\limits_{x\to 0}\dfrac{1-\cos x^3}{x^6}$

10. $\lim\limits_{x\to 0}\dfrac{1-\cos x^4}{x^8}$

11. $\lim\limits_{x\to\infty} x^{4/3}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right)$

12. $\lim\limits_{x\to\infty} x^{2/3}\left(\sqrt[3]{x+4}-\sqrt[3]{x-3}\right)$

In exercises 13 and 14, compare the limits to show that small errors can have disastrous effects.

13. $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x-1}$ and $\lim\limits_{x\to 1}\dfrac{x^2+x-2.01}{x-1}$

14. $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$ and $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4.01}$

15. Compare $f(x)=\sin\pi x$ and $g(x)=\sin 3.14x$ for $x=1$ (radian), $x=10$, $x=100$ and $x=1000$.

16. For exercise 1, follow the calculation of the function for $x=10^5$ as it would proceed for a machine computing with a 10-digit mantissa. Identify where the round-off error occurs.

In exercises 17 and 18, compare the exact answer to one obtained by a computer with a six-digit mantissa.

17. $(1.000003-1.000001)\times 10^7$

18. $(1.000006-1.000001)\times 10^7$

19. If you have access to a CAS, test it on the limits of examples 7.1, 7.6 and 7.7. Based on these results, do you think that your CAS does precise calculations or numerical estimates?

EXPLORATORY EXERCISES

1. In this exercise, we look at one aspect of the mathematical study of chaos. First, iterate the function $f(x)=x^2-2$ starting at $x_0=0.5$. That is, compute $x_1=f(0.5)$, then $x_2=f(x_1)$, then $x_3=f(x_2)$ and so on. Although the sequence of numbers stays bounded, the numbers never repeat (except by the accident of round-off errors). An impressive property of chaotic functions is the **butterfly effect** (more properly referred to as *sensitive dependence on initial conditions*). The butterfly effect applies to the chaotic nature of weather and states that the amount of air stirred by a butterfly flapping its wings in Brazil can create or disperse a tornado in Texas a few days later. Therefore, long-range weather prediction is impossible. To illustrate the butterfly effect, iterate $f(x)$ starting at $x_0=0.5$ and $x_0=0.51$. How many iterations does it take before the iterations are more than 0.1 apart? Try this again with $x_0=0.5$ and $x_0=0.501$. Repeat this exercise for the function $g(x)=x^2-1$. Even though the functions are almost identical, $g(x)$ is not chaotic and behaves very differently. This represents an important idea in modern medical research called dynamical diseases: a small change in

one of the constants in a function (e.g., the rate of an electrical signal within the human heart) can produce a dramatic change in the behavior of the system (e.g., the pumping of blood from the ventricles).

2. Just as we are subject to round-off error in using calculations from a computer, so are we subject to errors in a computer-generated graph. After all, the computer has to compute function values before it can decide where to plot points. On your computer or calculator, graph $y = \sin x^2$ (a disconnected graph—or point plot—is preferable). You should see the oscillations you expect from the sine function, but with the oscillations getting faster as x gets larger. Shift your graphing window to the right several times. At some point, the plot will become very messy and almost unreadable. Depending on your technology, you may see patterns in the plot. Are these patterns real or an illusion? To explain what is going on, recall that a computer graph is a finite set of pixels, with each pixel representing one x and one y. Suppose the computer is plotting points at $x = 0$, $x = 0.1$, $x = 0.2$ and so on. The y-values would then be $\sin 0^2$, $\sin 0.1^2$, $\sin 0.2^2$ and so on. Investigate what will happen between $x = 15$ and $x = 16$. Compute all the points $(15, \sin 15^2)$, $(15.1, \sin 15.1^2)$ and so on. If you were to graph these points, what pattern would emerge? To explain this pattern, argue that there is approximately half a period of the sine curve missing between each point plotted. Also, investigate what happens between $x = 31$ and $x = 32$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Secant line	Limit	Infinite limit
One-sided limit	Continuous	Loss-of-significance error
Removable discontinuity	Horizontal asymptote	Slant asymptote
Vertical asymptote	Squeeze Theorem	Intermediate Value Theorem
Method of bisections	Length of line segment	
Slope of curve		

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. In calculus, problems are often solved by first approximating the solution and then improving the approximation.
2. If $f(1.1) = 2.1$, $f(1.01) = 2.01$ and so on, then $\lim_{x\to 1} f(x) = 2$.
3. $\lim_{x\to a}[f(x)\cdot g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
4. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. If $f(2) = 1$ and $f(4) = 2$, then there exists an x between 2 and 4 such that $f(x) = 0$.
6. For any polynomial $p(x)$, $\lim_{x\to\infty} p(x) = \infty$.
7. If $f(x) = \frac{p(x)}{q(x)}$ for polynomials p and q with $q(a) = 0$, then f has a vertical asymptote at $x = a$.
8. Small round-off errors typically have only small effects on a calculation.
9. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a}\sqrt{f(x)} = \sqrt{L}$.

In exercises 1 and 2, numerically estimate the slope of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 - 2x$, $a = 2$
2. $f(x) = \sin 2x$, $a = 0$

In exercises 3 and 4, numerically estimate the length of the curve using (a) $n = 4$ and (b) $n = 8$ line segments and evenly spaced x-coordinates.

3. $f(x) = \sin x$, $0 \le x \le \frac{\pi}{4}$
4. $f(x) = x^2 - x$, $0 \le x \le 2$

In exercises 5–10, use numerical and graphical evidence to conjecture the value of the limit.

5. $\lim_{x\to 0}\frac{\tan^{-1} x^2}{x^2}$
6. $\lim_{x\to 1}\frac{x^2 - 1}{\ln x^2}$
7. $\lim_{x\to -2}\frac{x+2}{|x+2|}$
8. $\lim_{x\to 0}(1 + 2x)^{1/x}$
9. $\lim_{x\to\infty}\left(1 + \frac{2}{x}\right)^x$
10. $\lim_{x\to\infty} x^{2/x}$

Notice that the speed calculation in m/s is the same calculation we would use for the slope between the points (9.85, 100) and (19.79, 200). The connection between slope and speed (and other quantities of interest) is explored in this chapter.

2.1 TANGENT LINES AND VELOCITY

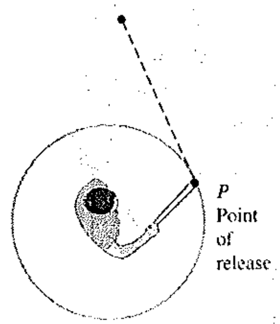


FIGURE 2.1
Path of rock

A traditional slingshot is essentially a rock on the end of a string, which you rotate around in a circular motion and then release. When you release the string, in which direction will the rock travel? An overhead view of this is illustrated in Figure 2.1. Many people mistakenly believe that the rock will follow a curved path, but Newton's first law of motion tells us that the path in the horizontal plane is straight. In fact, the rock follows a path along the tangent line to the circle at the point of release. Our aim in this section is to extend the notion of tangent line to more general curves.

To make our discussion more concrete, suppose that we want to find the tangent line to the curve $y = x^2 + 1$ at the point (1, 2) (see Figure 2.2). How could we define this? The tangent line hugs the curve near the point of tangency. In other words, like the tangent line to a circle, this tangent line has the same direction as the curve at the point of tangency. So, if you were standing on the curve at the point of tangency, took a small step and tried to stay on the curve, you would step in the direction of the tangent line. Another way to think of this is to observe that, if we zoom in sufficiently far, the graph appears to approximate that of a straight line. In Figure 2.3, we show the graph of $y = x^2 + 1$ zoomed in on the small rectangular box indicated in Figure 2.2. (Be aware that the "axes" indicated in Figure 2.3 do not intersect at the origin. We provide them only as a guide as to the scale used to produce the figure.) We now choose two points from the curve—for example, (1, 2) and (3, 10)—and compute the slope of the line joining these two points. Such a line is called a **secant** line and we denote its slope by m_{sec}:

$$m_{sec} = \frac{10-2}{3-1} = 4.$$

An equation of the secant line is then determined by

$$\frac{y-2}{x-1} = 4,$$

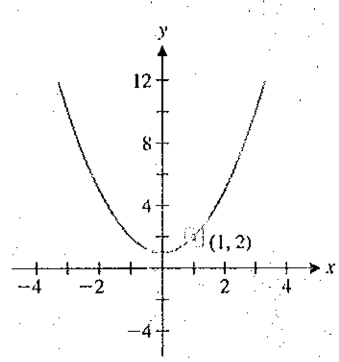


FIGURE 2.2
$y = x^2 + 1$

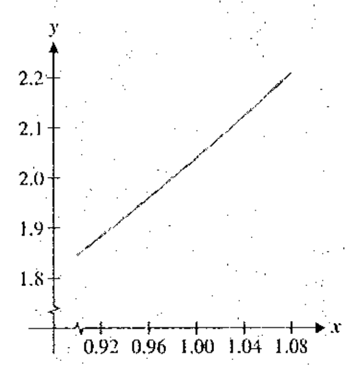


FIGURE 2.3
$y = x^2 + 1$

so that

$$y = 4(x - 1) + 2.$$

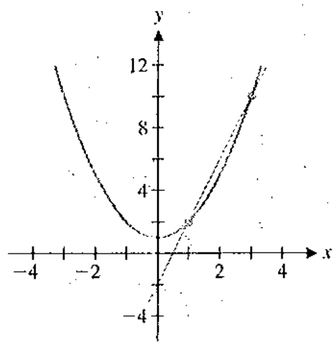


FIGURE 2.4a
Secant line joining (1, 2) and (3, 10)

As can be seen in Figure 2.4a, the secant line doesn't look very much like a tangent line.

Taking the second point a little closer to the point of tangency, say (2, 5), gives the slope of the secant line as

$$m_{\text{sec}} = \frac{5-2}{2-1} = 3$$

and an equation of the secant line as $y = 3(x - 1) + 2$. As seen in Figure 2.4b, this looks much more like a tangent line, but it's still not quite there. Choosing our second point much closer to the point of tangency, say (1.05, 2.1025), should give us an even better approximation to the tangent line. In this case, we have

$$m_{\text{sec}} = \frac{2.1025 - 2}{1.05 - 1} = 2.05$$

and an equation of the secant line is $y = 2.05(x - 1) + 2$. As can be seen in Figure 2.4c, the secant line looks very much like a tangent line, even when zoomed in quite far, as in Figure 2.4d. We continue this process by computing the slope of the secant line joining (1, 2) and the unspecified point $(1 + h, f(1 + h))$, for some value of h close to 0. The slope of this secant line is

$$\begin{aligned} m_{\text{sec}} &= \frac{f(1+h) - 2}{(1+h) - 1} = \frac{[(1+h)^2 + 1] - 2}{h} \\ &= \frac{(1 + 2h + h^2) - 1}{h} = \frac{2h + h^2}{h} && \text{Multiply out and cancel.} \\ &= \frac{h(2+h)}{h} = 2 + h. && \text{Factor out common } h \text{ and cancel.} \end{aligned}$$

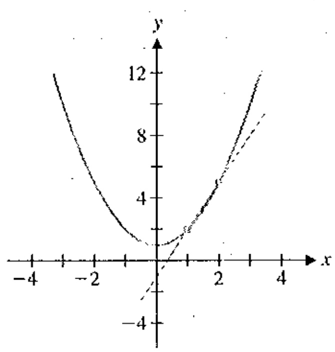


FIGURE 2.4b
Secant line joining (1, 2) and (2, 5)

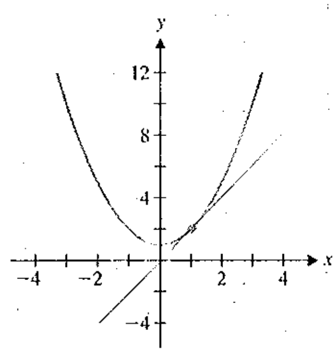


FIGURE 2.4c
Secant line joining (1, 2) and (1.05, 2.1025)

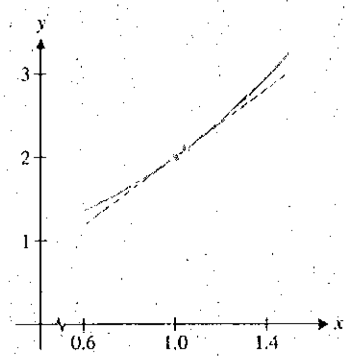


FIGURE 2.4d
Close-up of secant line

Notice that as h approaches 0, the slope of the secant line approaches 2, which we define to be the slope of the tangent line.

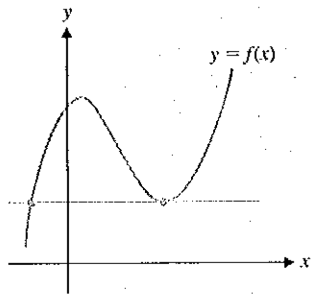


FIGURE 2.5
Tangent line intersecting a curve at more than one point

REMARK 1.1

We should make one more observation before moving on to the general case of tangent lines. Unlike the case for a circle, tangent lines may intersect a curve at more than one point, as indicated in Figure 2.5.

○ The General Case

To find the slope of the tangent line to $y = f(x)$ at $x = a$, first pick two points on the curve. One point is the point of tangency, $(a, f(a))$. Call the x-coordinate of the second point $x = a + h$, for some small number h; the corresponding y-coordinate is then $f(a + h)$. It is natural to think of h as being positive, as shown in Figure 2.6a, although h can also be negative, as shown in Figure 2.6b.

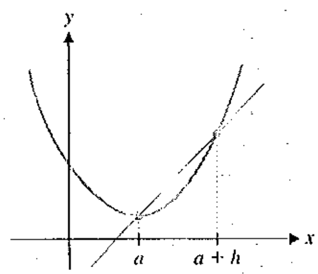


FIGURE 2.6a
Secant line ($h > 0$)

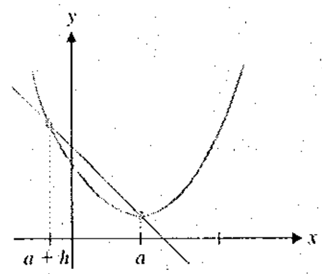


FIGURE 2.6b
Secant line ($h < 0$)

The slope of the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$ is given by

$$\boxed{m_{sec} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.} \tag{1.1}$$

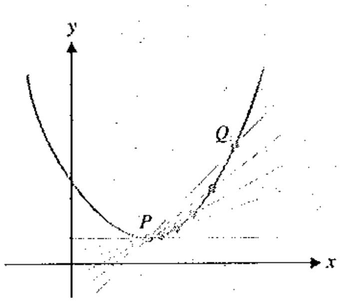


FIGURE 2.7
Secant lines approaching the tangent line at the point P

Notice that the expression in (1.1) (called a **difference quotient**) gives the slope of the secant line for any second point we might choose (i.e., for any $h \neq 0$). Recall that in order to obtain an improved approximation to the tangent line, we zoom in closer and closer toward the point of tangency. This makes the two points closer together, which in turn makes h closer to 0. Just how far should we zoom in? The farther, the better; this means that we want h to approach 0. We illustrate this process in Figure 2.7, where we have plotted a number of secant lines for $h > 0$. Notice that as the point Q approaches the point P (i.e., as $h \to 0$), the secant line approaches the tangent line at P.

We define the slope of the tangent line to be the limit of the slopes of the secant lines in (1.1) as h tends to 0, whenever this limit exists.

DEFINITION 1.1

The **slope** $m_{\tan}$ of the tangent line to $y = f(x)$ at $x = a$ is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{1.2}$$

provided the limit exists.

The tangent line is then the line passing through the point $(a, f(a))$ with slope $m_{\tan}$ and so, the point-slope form of the equation of the tangent line is

Equation of tangent line

$$\boxed{y = m_{\tan}(x - a) + f(a).}$$

EXAMPLE 1.1 Finding the Equation of a Tangent Line

Find an equation of the tangent line to $y = x^2 + 1$ at $x = 1$.

Solution We compute the slope using (1.2):

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{[(1+h)^2 + 1] - (1+1)}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 - 2}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} \\ &= \lim_{h \to 0} (2 + h) = 2. \end{aligned}$$

Multiply out and cancel.

Factor out common h and cancel.

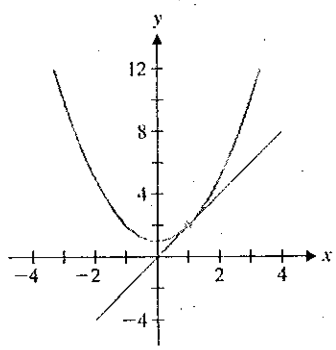


FIGURE 2.8
$y = x^2 + 1$ and the tangent line at $x = 1$

Notice that the point corresponding to $x = 1$ is $(1, 2)$ and the line with slope 2 through the point $(1, 2)$ has equation

$$y = 2(x - 1) + 2 \quad \text{or} \quad y = 2x.$$

Note how closely this corresponds to the secant lines computed earlier. We show a graph of the function and this tangent line in Figure 2.8. ■

EXAMPLE 1.2 Tangent Line to the Graph of a Rational Function

Find an equation of the tangent line to $y = \frac{2}{x}$ at $x = 2$.

Solution From (1.2), we have

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{h} \\ &= \lim_{h \to 0} \frac{\left[\frac{2 - (2+h)}{(2+h)}\right]}{h} = \lim_{h \to 0} \frac{\left[\frac{2 - 2 - h}{(2+h)}\right]}{h} \\ &= \lim_{h \to 0} \frac{-h}{(2+h)h} = \lim_{h \to 0} \frac{-1}{2+h} = -\frac{1}{2}. \end{aligned}$$

Since $f(2+h) = \frac{2}{2+h}$.

Add fractions and multiply out.

Cancel h's.

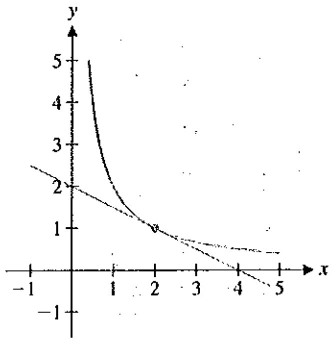


FIGURE 2.9
$y = \frac{2}{x}$ and tangent line at (2, 1)

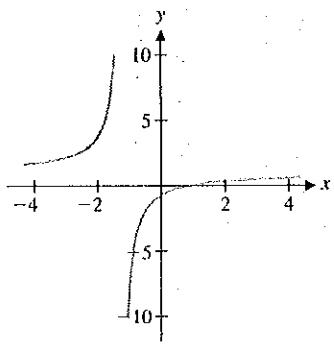


FIGURE 2.10a
$y = \frac{x-1}{x+1}$

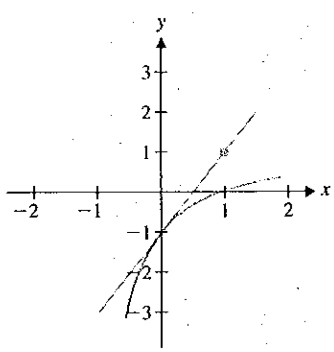


FIGURE 2.10b
Tangent line

The point corresponding to $x = 2$ is (2, 1), since $f(2) = 1$. An equation of the tangent line is then

$$y = -\frac{1}{2}(x - 2) + 1.$$

We show a graph of function and this tangent line in Figure 2.9. ■

In cases where we cannot (or cannot easily) evaluate the limit for the slope of the tangent line, we can approximate the limit numerically. We illustrate this in example 1.3.

EXAMPLE 1.3 Graphical and Numerical Approximation of Tangent Lines

Graphically and numerically approximate the slope of the tangent line to $y = \frac{x-1}{x+1}$ at $x = 0$.

Solution A graph of $y = \frac{x-1}{x+1}$ is shown in Figure 2.10a. We are interested in the tangent line at the point (0, −1). We sketch a tangent line in Figure 2.10b, where we have zoomed in to provide better detail. To approximate the slope, we estimate the coordinates of one point on the tangent line other than (0, −1). In Figure 2.10b, it appears that the tangent line passes through the point (1, 1). An estimate of the slope is then $m_{tan} \approx \frac{1-(-1)}{1-0} = 2$. To approximate the slope numerically, we choose several points near (0, −1) and compute the slopes of the secant lines. For example, rounding the y-values to four decimal places, we have

Second Point	m_{sec}	*Second Point*	m_{sec}
(1, 0)	$\frac{0-(-1)}{1-0} = 1$	(−0.5, −3)	$\frac{-3-(-1)}{-0.5-0} = 4.0$
(0.1, −0.8182)	$\frac{-0.8182-(-1)}{0.1-0} = 1.818$	(−0.1, −1.2222)	$\frac{-1.2222-(-1)}{-0.1-0} = 2.22$
(0.01, −0.9802)	$\frac{-0.9802-(-1)}{0.01-0} = 1.98$	(−0.01, −1.0202)	$\frac{-1.0202-(-1)}{-0.01-0} = 2.02$

In both columns, as the second point gets closer to (0, −1), the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point (0, −1) is then 2. ■

○ Velocity

The slopes of tangent lines have many important applications, of which one of the most important is in computing velocity. The term *velocity* is certainly familiar to you, but can you say precisely what it is? We often describe velocity as a quantity determining the speed *and direction* of an object, but what exactly is speed? If your car did not have a speedometer, you might determine your speed using the familiar formula

$$\text{distance} = \text{rate} \times \text{time}. \tag{1.3}$$

Using (1.3), you can find the rate (speed) by simply dividing the distance by the time. However, the rate in (1.3) refers to *average* speed over a period of time. We are interested in the speed at a specific instant. The following story should indicate the difference.

During traffic stops, police officers frequently ask drivers if they know how fast they were going. An overzealous student might answer that during the past, say, 3 years, 2 months, 7 days, 5 hours and 45 minutes, they've driven exactly 45,259.7 miles, so that their speed was

$$\text{rate} = \frac{\text{distance}}{\text{time}} = \frac{45{,}259.7 \text{ miles}}{27{,}917.75 \text{ hours}} \approx 1.62118 \text{ mph.}$$

Of course, most police officers would not be impressed with this analysis, but, *why* is it wrong? Certainly there's nothing wrong with formula (1.3) or the arithmetic. However, it's reasonable to argue that unless they were in their car during this entire 3-year period, the results are invalid.

Suppose that the driver substitutes the following argument instead: "I left home at 6:17 P.M. and by the time you pulled me over at 6:43 P.M., I had driven exactly 17 miles. Therefore, my speed was

$$\text{rate} = \frac{17 \text{ miles}}{26 \text{ minutes}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = 39.2 \text{ mph,}$$

well under the posted 45-mph speed limit."

While this is a much better estimate of the velocity than the 1.6 mph computed previously, it's still an average velocity using too long of a time period.

More generally, suppose that the function $f(t)$ gives the position at time t of an object moving along a straight line. That is, $f(t)$ gives the displacement (**signed distance**) from a fixed reference point, so that $f(t) < 0$ means that the object is located $|f(t)|$ away from the reference point, but in the negative direction. Then, for two times a and b (where $a < b$), $f(b) - f(a)$ gives the signed distance between positions $f(a)$ and $f(b)$. The **average velocity** v_{avg} is then given by

$$v_{avg} = \frac{\text{signed distance}}{\text{time}} = \frac{f(b) - f(a)}{b - a}. \tag{1.4}$$

EXAMPLE 1.4 Finding Average Velocity

The position of a car after t minutes driving in a straight line is given by

$$s(t) = \frac{1}{2}t^2 - \frac{1}{12}t^3, \quad 0 \le t \le 4.$$

Approximate the velocity at time $t = 2$.

Solution Averaging over the 2 minutes from $t = 2$ to $t = 4$, we get from (1.4) that

$$\begin{aligned} v_{avg} &= \frac{s(4) - s(2)}{4 - 2} \approx \frac{2.666666667 - 1.333333333}{2} \\ &\approx 0.666666667 \text{ mile/minute} \\ &\approx 40 \text{ mph.} \end{aligned}$$

Of course, a 2-minute-long interval is rather long, given that cars can speed up and slow down a great deal in 2 minutes. We get an improved approximation by averaging over just one minute:

$$\begin{aligned} v_{avg} &= \frac{s(3) - s(2)}{3 - 2} \approx \frac{2.25 - 1.333333333}{1} \\ &\approx 0.9166666667 \text{ mile/minute} \\ &\approx 55 \text{ mph.} \end{aligned}$$

h	$\frac{s(2+h)-s(2)}{h}$
1.0	0.9166666667
0.1	0.9991666667
0.01	0.9999916667
0.001	0.999999917
0.0001	1.0
0.00001	1.0

While this latest estimate is certainly better than the first one, we can do better. As we make the time interval shorter and shorter, the average velocity should be getting closer and closer to the velocity at the instant $t = 2$. It stands to reason that, if we compute the average velocity over the time interval $[2, 2+h]$ and then let $h \to 0$, the resulting average velocities should be getting closer and closer to the velocity at the instant $t = 2$.

We have
$$v_{avg} = \frac{s(2+h)-s(2)}{(2+h)-2} = \frac{s(2+h)-s(2)}{h}.$$

A sequence of these average velocities is displayed in the accompanying table, for $h > 0$, with similar results if we allow h to be negative. It appears that the average velocity is approaching 1 mile/minute (60 mph), as $h \to 0$. We refer to this limiting value as the *instantaneous velocity*.

This leads us to make the following definition.

NOTE

Notice that if (for example) t is measured in seconds and $f(t)$ is measured in feet, then velocity (average or instantaneous) is measured in feet per second (ft/s). The term velocity is always used to refer to instantaneous velocity.

DEFINITION 1.2

If $f(t)$ represents the position of an object relative to some fixed location at time t as it moves along a straight line, then the **instantaneous velocity** at time $t = a$ is given by

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \tag{1.5}$$

provided the limit exists.

EXAMPLE 1.5 Finding Average and Instantaneous Velocity

Suppose that the height of a falling object t seconds after being dropped from a height of 64 feet is given by $f(t) = 64 - 16t^2$ feet. Find the average velocity between times $t = 1$ and $t = 2$; the average velocity between times $t = 1.5$ and $t = 2$; the average velocity between times $t = 1.9$ and $t = 2$ and the instantaneous velocity at time $t = 2$.

Solution The average velocity between times $t = 1$ and $t = 2$ is

$$v_{avg} = \frac{f(2)-f(1)}{2-1} = \frac{64-16(2)^2-[64-16(1)^2]}{1} = -48 \text{ ft/s}.$$

The average velocity between times $t = 1.5$ and $t = 2$ is

$$v_{avg} = \frac{f(2)-f(1.5)}{2-1.5} = \frac{64-16(2)^2-[64-16(1.5)^2]}{0.5} = -56 \text{ ft/s}.$$

The average velocity between times $t = 1.9$ and $t = 2$ is

$$v_{avg} = \frac{f(2)-f(1.9)}{2-1.9} = \frac{64-16(2)^2-[64-16(1.9)^2]}{0.1} = -62.4 \text{ ft/s}.$$

The instantaneous velocity is the limit of such average velocities. From (1.5), we have

$$\begin{aligned} v(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{(2+h)-2} \\ &= \lim_{h\to 0} \frac{[64-16(2+h)^2]-[64-16(2)^2]}{h} \\ &= \lim_{h\to 0} \frac{[64-16(4+4h+h^2)]-[64-16(2)^2]}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h\to 0} \frac{-64h-16h^2}{h} = \lim_{h\to 0} \frac{-16h(h+4)}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h\to 0}[-16(h+4)] = -64 \text{ ft/s}. \end{aligned}$$

Recall that velocity indicates both speed and direction. In this problem, $f(t)$ measures the height above the ground. So, the negative velocity indicates that the object is moving in the negative (or downward) direction. The **speed** of the object at the 2-second mark is then 64 ft/s. (Speed is simply the absolute value of velocity.)

Observe that the formulas for instantaneous velocity (1.5) and for the slope of a tangent line (1.2) are identical. We want to make this connection as strong as possible, by illustrating example 1.5 graphically. We graph the position function $f(t) = 64 - 16t^2$ for $0 \le t \le 2$. The average velocity between $t = 1$ and $t = 2$ corresponds to the slope of the secant line between the points at $t = 1$ and $t = 2$. (See Figure 2.11a.) Similarly, the average velocity between $t = 1.5$ and $t = 2$ gives the slope of the corresponding secant line. (See Figure 2.11b.) Finally, the instantaneous velocity at time $t = 2$ corresponds to the slope of the tangent line at $t = 2$. (See Figure 2.11c.)

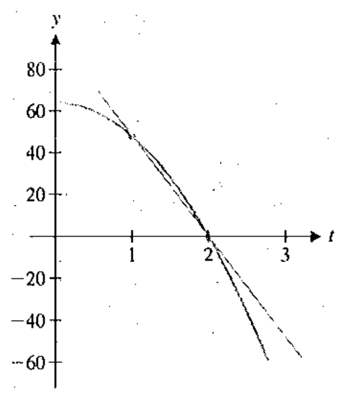


FIGURE 2.11a
Secant line between $t = 1$ and $t = 2$

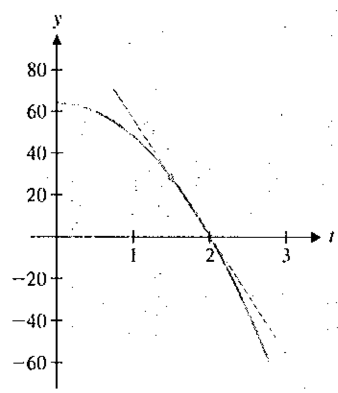


FIGURE 2.11b
Secant line between $t = 1.5$ and $t = 2$

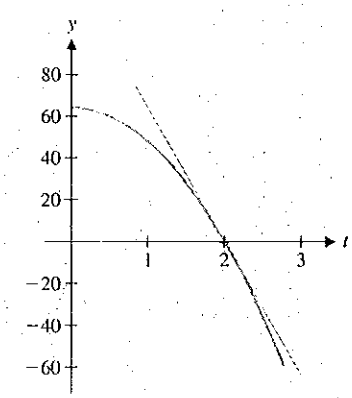


FIGURE 2.11c
Tangent line at $t = 2$

Velocity is a *rate* (more precisely, the instantaneous rate of change of position with respect to time). We now generalize this notion of instantaneous rate of change. In general, the **average rate of change** of a function $f(x)$ between $x = a$ and $x = b\ (a \neq b)$ is given by

$$\frac{f(b) - f(a)}{b - a}.$$

The **instantaneous rate of change** of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. The units of the instantaneous rate of change are the units of f divided by (or "per") the units of x.

EXAMPLE 1.6 Interpreting Rates of Change

If the function $f(t)$ gives the population of a city in millions of people t years after January 1, 2000, interpret each of the following quantities, assuming that they equal the given numbers. (a) $\frac{f(2) - f(0)}{2} = 0.34$, (b) $f(2) - f(1) = 0.31$ and (c) $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 0.3$.

Solution From the preceding, $\frac{f(b) - f(a)}{b - a}$ is the average rate of change of the function f between a and b. Expression (a) tells us that the average rate of change of f between $a = 0$ and $b = 2$ is 0.34. Stated in more common language, the city's population grew at an average rate of 0.34 million people per year between 2000 and 2002. Similarly, expression (b) is the average rate of change between $a = 1$ and $b = 2$. That is, the city's population grew at an average rate of 0.31 million people per year in 2001. Finally, expression (c) gives the instantaneous rate of change of the population at time $t = 2$. As of January 1, 2002, the city's population was growing at a rate of 0.3 million people per year. ■

Additional applications of the slope of a tangent line are innumerable. These include the rate of a chemical reaction, the inflation rate in economics and learning growth rates in psychology. Rates of change in nearly any discipline you can name can be thought of as slopes of tangent lines. We explore many of these applications as we progress through the text.

You hopefully noticed that we tacked the phrase "provided the limit exists" onto the end of the definitions of the slope of a tangent line, the instantaneous velocity and the instantaneous rate of change. This was important, since these defining limits do not always exist, as we see in example 1.7.

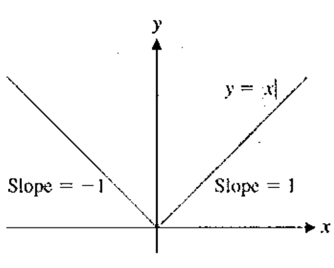


FIGURE 2.12
$y = |x|$

EXAMPLE 1.7 A Graph with No Tangent Line at a Point

Determine whether there is a tangent line to $y = |x|$ at $x = 0$.

Solution We can look at this problem graphically, numerically and symbolically. The graph is shown in Figure 2.12. Our graphical technique is to zoom in on the point of tangency until the graph appears straight. However, no matter how far we zoom in on (0, 0), the graph continues to look like Figure 2.12. (This is one reason why we left off the scale on Figure 2.12.) From this evidence alone, we would conjecture that the tangent line does not exist. Numerically, the slope of the tangent line is the limit of the slope of a secant line, as the second point approaches the point of tangency. Observe that the secant line through (0, 0) and (1, 1) has slope 1, as does the secant line through (0, 0) and (0.1, 0.1). In fact, if h is any positive number, the slope of the secant line through (0, 0) and $(h, |h|)$ is 1. However, the secant line through (0, 0) and $(-1, 1)$ has slope -1, as does the secant line through (0, 0) and $(h, |h|)$ for any negative number h. We therefore conjecture that the one-sided limits are different, so that the limit (and also the tangent line) does not exist. To prove this conjecture, we take our cue from the numerical work and look at one-sided limits: if $h > 0$, then $|h| = h$, so that

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

On the other hand, if $h < 0$, then $|h| = -h$ (remember that if $h < 0$, $-h > 0$), so that

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

Since the one-sided limits are different, we conclude that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \text{ does not exist}$$

and hence, the tangent line does not exist. ■

EXERCISES 2.1

WRITING EXERCISES

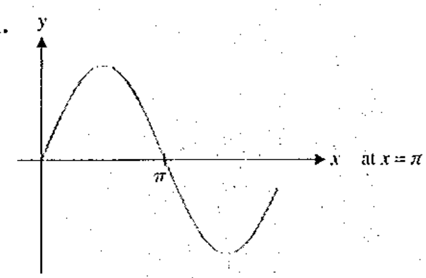
1. What does the phrase "off on a tangent" mean? Relate the common meaning of the phrase to the image of a tangent to a circle (use the slingshot example, if that helps). In what way does the zoomed image of the tangent promote the opposite view of the relationship between a curve and its tangent?

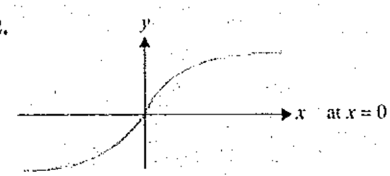
2. In general, the instantaneous velocity of an object cannot be computed directly; the limit process is the only way to compute velocity *at an instant*. Given this, how does a car's speedometer compute velocity? (Hint: Look this up in a reference book or on the Internet. An important aspect of the car's ability to do this seemingly difficult task is that it performs *analog* calculations. For example, the pitch of a fly's buzz gives us an analog device for computing the speed of a fly's wings, since pitch is proportional to speed.)

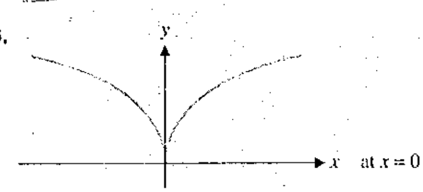
3. Look in the news media (TV, newspaper, Internet) and find references to at least five different *rates*. We have defined a rate of change as the limit of the difference quotient of a function. For your five examples, state as precisely as possible what the original function is. Is the rate given quantitatively or qualitatively? If it is given quantitatively, is the rate given as a percentage or a number? In calculus, we usually compute rates (quantitatively) as numbers; is this in line with the standard usage?

4. Sketch the graph of a function that is discontinuous at $x = 1$. Explain why there is no tangent line at $x = 1$.

In exercises 1–4, sketch in a plausible tangent line at the given point. (Hint: Mentally zoom in on the point and use the zoomed image of the tangent.)

1. 

2. 

3. 

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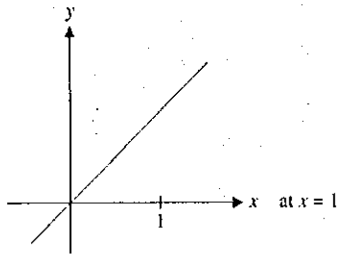
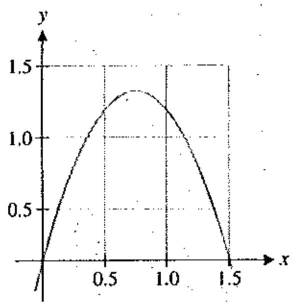
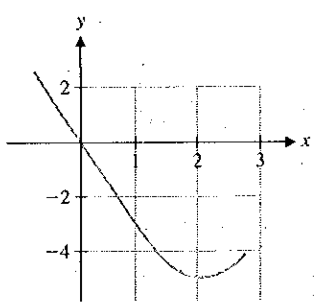


In exercises 5 and 6, estimate the slope of the tangent line to the curve at $x = 1$.

5.

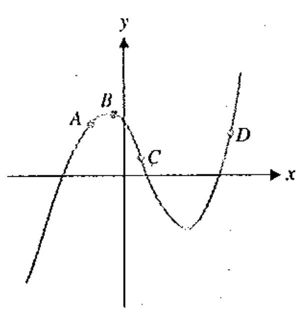


6.

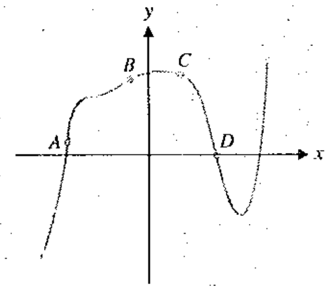


In exercises 7 and 8, list the points A, B, C and D in order of increasing slope of the tangent line.

7.



8.



In exercises 9–12, compute the slope of the secant line between the points at (a) $x = 1$ and $x = 2$, (b) $x = 2$ and $x = 3$, (c) $x = 1.5$ and $x = 2$, (d) $x = 2$ and $x = 2.5$, (e) $x = 1.9$ and $x = 2$, (f) $x = 2$ and $x = 2.1$, and (g) use parts (a)–(f) and other calculations as needed to estimate the slope of the tangent line at $x = 2$.

9. $f(x) = x^3 - x$
10. $f(x) = \sqrt{x^2 + 1}$
11. $f(x) = \cos x^2$
12. $f(x) = \tan(x/4)$

In exercises 13–16, use a CAS or graphing calculator.

13. On one graph, sketch the secant lines in exercise 9, parts (a)–(d) and the tangent line in part (g).
14. On one graph, sketch the secant lines in exercise 10, parts (a)–(d) and the tangent line in part (g).
15. Animate the secant lines in exercise 9, parts (a), (c) and (e), converging to the tangent line in part (g).
16. Animate the secant lines in exercise 9, parts (b), (d) and (f), converging to the tangent line in part (g).

In exercises 17–24, find the equation of the tangent line to $y = f(x)$ at $x = a$. Graph $y = f(x)$ and the tangent line to verify that you have the correct equation.

17. $f(x) = x^2 - 2,\ a = 1$
18. $f(x) = x^2 - 2,\ a = 0$
19. $f(x) = x^2 - 3x,\ a = -2$
20. $f(x) = x^3 + x,\ a = 1$
21. $f(x) = \dfrac{2}{x+1},\ a = 1$
22. $f(x) = \dfrac{x}{x-1},\ a = 0$
23. $f(x) = \sqrt{x+3},\ a = -2$
24. $f(x) = \sqrt{x^2+1},\ a = 1$

In exercises 25–30, use graphical and numerical evidence to determine whether the tangent line to $y = f(x)$ exists at $x = a$. If it does, estimate the slope of the tangent; if not, explain why not.

25. $f(x) = |x - 1|$ at $a = 1$
26. $f(x) = \dfrac{4x}{x-1}$ at $a = 1$
27. $f(x) = \begin{cases} -2x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \ge 0 \end{cases}$ at $a = 0$

28. $f(x) = \begin{cases} -2x & \text{if } x < 1 \\ x - 3 & \text{if } x \geq 1 \end{cases}$ at $a = 1$

29. $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ x^3 + 1 & \text{if } x \geq 0 \end{cases}$ at $a = 0$

30. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x^2 - 2x & \text{if } x > 0 \end{cases}$ at $a = 0$

In exercises 31–34, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

31. $f(t) = 16t^2 + 10$

32. $f(t) = 3t^3 + t$

33. $f(t) = \sqrt{t^2 + 8t}$

34. $f(t) = 100\sin(t/4)$

In exercises 35 and 36, use the position function $f(t)$ meters to find the velocity at time $t = a$ seconds.

35. $f(t) = -16t^2 + 5$, (a) $a = 1$; (b) $a = 2$

36. $f(t) = \sqrt{t + 16}$, (a) $a = 0$; (b) $a = 2$

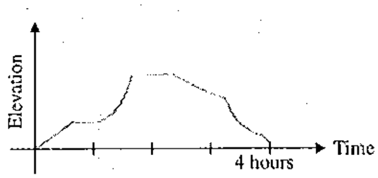
37. The table shows the freezing temperature of water in degrees Celsius at various pressures. Estimate the slope of the tangent line at $p = 1$ and interpret the result. Estimate the slope of the tangent line at $p = 3$ and interpret the result.

p (atm)	0	1	2	3	4
°C	0	−7	−20	−16	−11

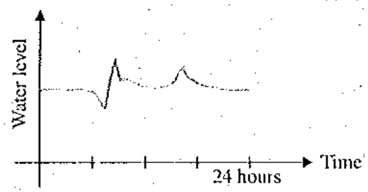
38. The table shows the range of a soccer kick launched at 30° above the horizontal at various initial speeds. Estimate the slope of the tangent line at $v = 50$ and interpret the result.

Distance (yd)	19	28	37	47	58
Speed (mph)	30	40	50	60	70

39. The graph shows the elevation of a person on a hike up a mountain as a function of time. When did the hiker reach the top? When was the hiker going the fastest on the way up? When was the hiker going the fastest on the way down? What do you think occurred at places where the graph is level?



40. The graph shows the amount of water in a city water tank as a function of time. When was the tank the fullest? the emptiest? When was the tank filling up at the fastest rate? When was the tank emptying at the fastest rate? What time of day do you think the level portion represents?



41. Suppose a hot cup of coffee is left in a room for 2 hours. Sketch a reasonable graph of what the temperature would look like as a function of time. Then sketch a graph of what the rate of change of the temperature would look like.

42. Sketch a graph representing the height of a bungee-jumper. Sketch the graph of the person's velocity (use + for upward velocity and − for downward velocity).

43. Suppose that $f(t)$ represents the balance in dollars of a bank account t years after January 1, 2000. Interpret each of the following. (a) $\frac{f(4) - f(2)}{2} = 21{,}034$, (b) $2[f(4) - f(3.5)] = 25{,}036$ and (c) $\lim_{h \to 0} \frac{f(4 + h) - f(4)}{h} = 30{,}000$.

44. Suppose that $f(m)$ represents the value of a car that has been driven m thousand miles. Interpret each of the following. (a) $\frac{f(40) - f(38)}{2} = -2103$, (b) $f(40) - f(39) = -2040$ and (c) $\lim_{h \to 0} \frac{f(40 + h) - f(40)}{h} = -2000$.

45. In using a slingshot, it is important to generate a large angular velocity. **Angular velocity** is defined by $\lim_{h \to 0} \frac{\theta(a + h) - \theta(a)}{h}$, where $\theta(t)$ is the angle of rotation at time t. If the angle of a slingshot is $\theta(t) = 0.4t^2$, what is the angular velocity after three rotations? [Hint: Which value of t (seconds) corresponds to three rotations?]

46. Find the angular velocity of the slingshot in exercise 45 after two rotations. Explain why the third rotation is helpful.

47. Sometimes an incorrect method accidentally produces a correct answer. For quadratic functions (but definitely *not* most other functions), the average velocity between $t = r$ and $t = s$ equals the average of the velocities at $t = r$ and $t = s$. To show this, assume that $f(t) = at^2 + bt + c$ is the distance function. Show that the average velocity between $t = r$ and $t = s$ equals $a(s + r) + b$. Show that the velocity at $t = r$ is $2ar + b$ and the velocity at $t = s$ is $2as + b$. Finally, show that $a(s + r) + b = \frac{(2ar + b) + (2as + b)}{2}$.

48. Find a cubic function [try $f(t) = t^3 + \cdots$] and numbers r and s such that the average velocity between $t = r$ and $t = s$ is different from the average of the velocities at $t = r$ and $t = s$.

49. Show that $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. (Hint: Let $h = x - a$.)

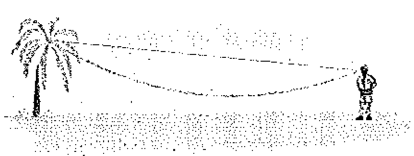
50. Use the second limit in exercise 49 to recompute the slope in exercises 17 and 19. Which limit do you prefer?

51. A car speeding around a curve in the shape of $y = x^2$ (moving from left to right) skids off at the point $(\frac{1}{2}, \frac{1}{4})$. If the car continues in a straight path, will it hit a tree located at the point $(1, \frac{3}{4})$?

52. For the car in exercise 51, show graphically that there is only one skid point on the curve $y = x^2$ such that the tangent line passes through the point $(1, \frac{3}{4})$.

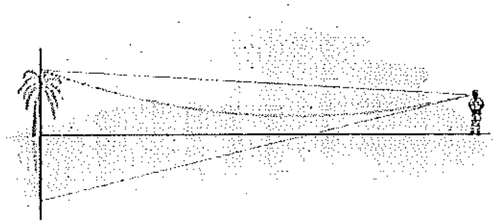
EXPLORATORY EXERCISES

1. Many optical illusions are caused by our brain's (unconscious) use of the tangent line in determining the positions of objects. Suppose you are in the desert 100 feet from a palm tree. You see a particular spot 10 feet up on the palm tree due to light reflecting from that spot to your eyes. Normally, it is a good approximation to say that the light follows a straight line (top path in the figure).



Two paths of light from tree to person.

However, when there is a large temperature difference in the air, light may follow nonlinear paths. If, as in the desert, the air near the ground is much hotter than the air higher up, light will bend as indicated by the bottom path in the figure. Our brains always interpret light coming in straight paths, so you would think the spot on the tree is at $y = 10$ because of the top path and also at some other y because of the bottom path. If the bottom curve is $y = 0.002x^2 - 0.24x + 10$, find an equation of the tangent line at $x = 100$ and show that it crosses the y-axis at $y = -10$. That is, you would "see" the spot at $y = 10$ and also at $y = -10$, a perfect reflection.



Two perceived locations of tree.

How do reflections normally occur in nature? From water! You would perceive a tree and its reflection in a pool of water. This is the desert mirage!

2. You can use a VCR to estimate speed. Most VCRs play at 30 frames per second. So, with a frame-by-frame advance, you can estimate time as the number of frames divided by 30. If you know the distance covered, you can compute the average velocity by dividing distance by time. Try this to estimate how fast you can throw a ball, run 50 yards, hit a tennis ball or whatever speed you find interesting. Some of the possible inaccuracies are explored in exercise 3.

3. What is the peak speed for a human being? It has been estimated that Carl Lewis reached a peak speed of 28 mph while winning a gold medal in the 1992 Olympics. Suppose that we have the following data for a sprinter.

Meters	*Seconds*
30	3.2
40	4.2
50	5.16666
56	5.76666
58	5.93333
60	6.1

Meters	*Seconds*
62	6.26666
64	6.46666
70	7.06666
80	8.0
90	9.0
100	10.0

We want to estimate peak speed. We could start by computing $\frac{\text{distance}}{\text{time}} = \frac{100 \text{ m}}{10 \text{ s}} = 10$ m/s, but this is the average speed over the entire race, not the peak speed. Argue that we want to compute average speeds only using adjacent measurements (e.g., 40 and 50 meters, or 50 and 56 meters). Do this for all 11 adjacent pairs and find the largest speed (if you want to convert to mph, divide by 0.447). We will then explore how accurate this estimate might be.

Notice that all times are essentially multiples of 1/30, since the data were obtained using the VCR technique in exercise 2. Given this, why is it suspicious that all the distances are whole numbers? To get an idea of how much this might affect your calculations, change some of the distances. For instance, if you change 60 (meters) to 59.8, how much do your average velocity calculations change? One possible way to identify where mistakes have been made is to look at the pattern of average velocities: does it seem reasonable? Would a sprinter speed up and slow down in such a pattern? In places where the pattern seems suspicious, try adjusting the distances and see if you can produce a more realistic pattern. Taking all this into account, try to quantify your error analysis: what is the highest (lowest) the peak speed could be?

2.2 THE DERIVATIVE

In section 2.1, we investigated two seemingly unrelated concepts: slopes of tangent lines and velocity, both of which are expressed in terms of the *same* limit. This is an indication of the power of mathematics, that otherwise unrelated notions are described by the *same* mathematical expression. This particular limit turns out to be so useful that we give it a special name.

DEFINITION 2.1

The **derivative** of the function $f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{2.1}$$

provided the limit exists. If the limit exists, we say that f is **differentiable** at $x = a$.

An alternative form of (2.1) is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}. \tag{2.2}$$

(See exercise 49 in section 2.1.)

EXAMPLE 2.1 Finding the Derivative at a Point

Compute the derivative of $f(x) = 3x^3 + 2x - 1$ at $x = 1$.

Solution From (2.1), we have

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{[3(1+h)^3 + 2(1+h) - 1] - (3 + 2 - 1)}{h} \\ &= \lim_{h \to 0} \frac{3(1 + 3h + 3h^2 + h^3) + (2 + 2h) - 1 - 4}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h \to 0} \frac{11h + 9h^2 + 3h^3}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h \to 0} (11 + 9h + 3h^2) = 11. \end{aligned}$$

Suppose that in example 2.1 we had also needed to find $f'(2)$ and $f'(3)$. Must we now repeat the same long limit calculation to find each of $f'(2)$ and $f'(3)$? Instead, we compute the derivative without specifying a value for x, leaving us with a function from which we can calculate $f'(a)$ for any a, simply by substituting a for x.

EXAMPLE 2.2 Finding the Derivative at an Unspecified Point

Find the derivative of $f(x) = 3x^3 + 2x - 1$ at an unspecified value of x. Then, evaluate the derivative at $x = 1$, $x = 2$ and $x = 3$.

Solution Replacing a with x in the definition of the derivative (2.1), we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left[3(x+h)^3 + 2(x+h) - 1\right] - (3x^3 + 2x - 1)}{h} \\
&= \lim_{h\to 0} \frac{3(x^3 + 3x^2h + 3xh^2 + h^3) + (2x + 2h) - 1 - 3x^3 - 2x + 1}{h} && \text{Multiply out and cancel.} \\
&= \lim_{h\to 0} \frac{9x^2h + 9xh^2 + 3h^3 + 2h}{h} && \text{Factor out common } h \text{ and cancel.} \\
&= \lim_{h\to 0} (9x^2 + 9xh + 3h^2 + 2) \\
&= 9x^2 + 0 + 0 + 2 = 9x^2 + 2.
\end{aligned}$$

Notice that in this case, we have derived a new *function*, $f'(x) = 9x^2 + 2$. Simply substituting in for x, we get $f'(1) = 9 + 2 = 11$ (the same as we got in example 2.1!), $f'(2) = 9(4) + 2 = 38$ and $f'(3) = 9(9) + 2 = 83$. ■

Example 2.2 leads us to the following definition.

DEFINITION 2.2

The **derivative** of $f(x)$ is the function $f'(x)$ given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{2.3}$$

provided the limit exists. The process of computing a derivative is called **differentiation.**

Further, f is differentiable on an interval I if it is differentiable at every point in I.

In examples 2.3 and 2.4, observe that the name of the game is to write down the defining limit and then to find some way of evaluating that limit (which initially has the indeterminate form $\frac{0}{0}$).

EXAMPLE 2.3 Finding the Derivative of a Simple Rational Function

If $f(x) = \dfrac{1}{x}$ $(x \neq 0)$, find $f'(x)$.

Solution We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)}{h} && \text{Since } f(x+h) = \frac{1}{x+h}. \\
&= \lim_{h\to 0} \frac{\left[\dfrac{x - (x+h)}{x(x+h)}\right]}{h} && \text{Add fractions and cancel.} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} && \text{Cancel } h\text{'s.} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2},
\end{aligned}$$

or $f'(x) = -x^{-2}$. ■

EXAMPLE 2.4 **The Derivative of the Square Root Function**

If $f(x) = \sqrt{x}$ (for $x \geq 0$), find $f'(x)$.

Solution We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) && \text{Multiply numerator and denominator by the conjugate: } \sqrt{x+h} + \sqrt{x}. \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h\left[\sqrt{x+h} + \sqrt{x}\right]} && \text{Multiply out and cancel.} \\ &= \lim_{h\to 0} \frac{h}{h\left[\sqrt{x+h} + \sqrt{x}\right]} && \text{Cancel common } h\text{'s.} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}. \end{aligned}$$

Notice that $f'(x)$ is defined only for $x > 0$. ■

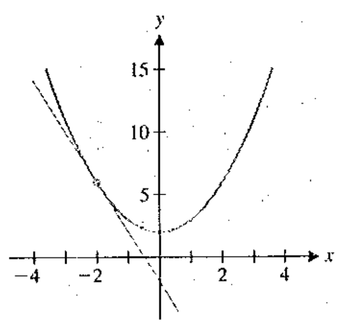


FIGURE 2.13a
$m_{tan} < 0$

The benefits of having a derivative *function* go well beyond simplifying the computation of a derivative at multiple points. As we'll see, the derivative function tells us a great deal about the original function.

Keep in mind that the value of a derivative at a point is the slope of the tangent line at that point. In Figures 2.13a–2.13c, we have graphed a function along with its tangent lines at three different points. The slope of the tangent line in Figure 2.13a is negative; the slope of the tangent line in Figure 2.13c is positive and the slope of the tangent line in Figure 2.13b is zero. These three tangent lines give us three points on the graph of the derivative function (see Figure 2.13d), by estimating the value of $f'(x)$ at the three points. Thus, as x changes, the slope of the tangent line changes and hence $f'(x)$ changes.

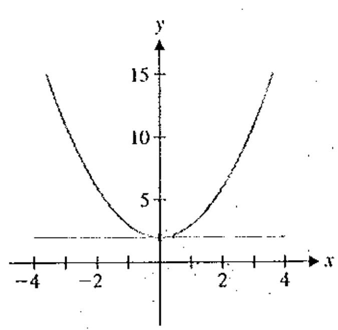


FIGURE 2.13b
$m_{tan} = 0$

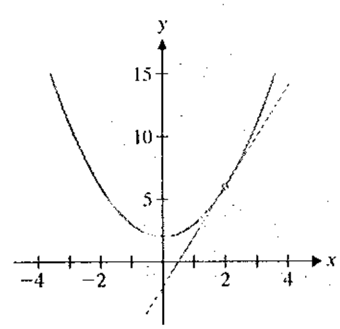


FIGURE 2.13c
$m_{tan} > 0$

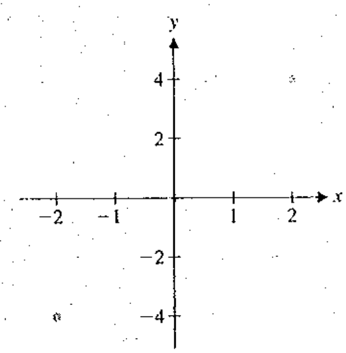


FIGURE 2.13d
$y = f'(x)$ (three points)

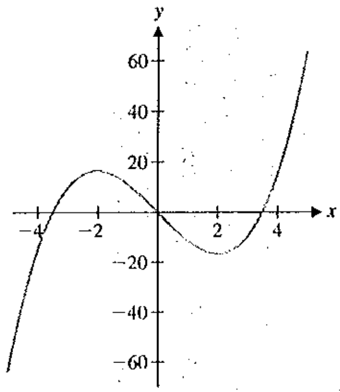


FIGURE 2.14
$y = f(x)$

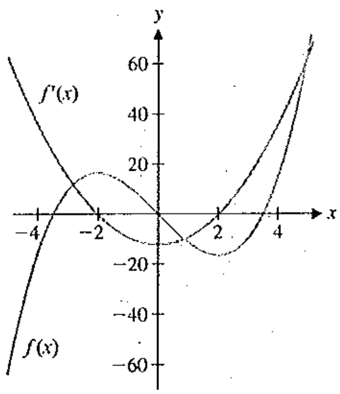


FIGURE 2.15
$y = f(x)$ and $y = f'(x)$

EXAMPLE 2.5 Sketching the Graph of $f'(x)$ Given the Graph of $f(x)$

Given the graph of $f(x)$ in Figure 2.14, sketch a plausible graph of $f'(x)$.

Solution Rather than worrying about exact values of the slope, we only wish to get the general shape right. As in Figures 2.13a–2.13d, pick a few important points to analyze carefully. You should focus on any discontinuities and any places where the graph of f turns around.

The graph levels out at approximately $x = -2$ and $x = 2$. At these points, the derivative is 0. As we move from left to right, the graph rises for $x < -2$, drops for $-2 < x < 2$ and rises again for $x > 2$. This means that $f'(x) > 0$ for $x < -2$, $f'(x) < 0$ for $-2 < x < 2$ and finally $f'(x) > 0$ for $x > 2$. We can say even more. As x approaches -2 from the left, observe that the tangent lines get less steep. Therefore, $f'(x)$ becomes less positive as x approaches -2 from the left. Moving to the right from $x = -2$, the graph gets steeper until about $x = 0$, then gets less steep until it levels out at $x = 2$. Thus, $f'(x)$ gets more negative until $x = 0$, then less negative until $x = 2$. Finally, the graph gets steeper as we move to the right from $x = 2$. Putting this all together, we have the possible graph of $f'(x)$ shown in red in Figure 2.15, superimposed on the graph of $f(x)$. ∎

The opposite question to that asked in example 2.5 is even more interesting. That is, given the graph of a derivative, what might the graph of the original function look like? We explore this in example 2.6.

EXAMPLE 2.6 Sketching the Graph of $f(x)$ Given the Graph of $f'(x)$

Given the graph of $f'(x)$ in Figure 2.16, sketch a plausible graph of $f(x)$.

Solution Again, do not worry about getting exact values of the function, but rather only the general shape of the graph. Notice from the graph of $y = f'(x)$ that $f'(x) < 0$ for $x < -2$, so that on this interval, the slopes of the tangent lines to $y = f(x)$ are negative and the function is decreasing. On the interval $(-2, 1)$, $f'(x) > 0$, indicating that the tangent lines to the graph of $y = f(x)$ have positive slope and the function is increasing. Further, this says that the graph turns around (i.e., goes from decreasing to increasing) at $x = -2$. We have drawn a graph exhibiting this behavior in Figure 2.17

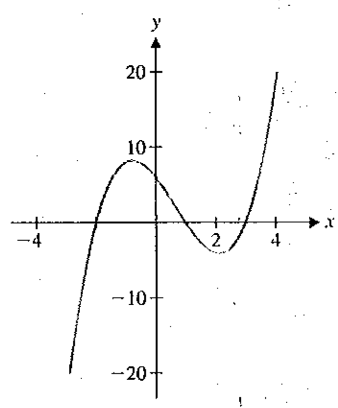


FIGURE 2.16
$y = f'(x)$

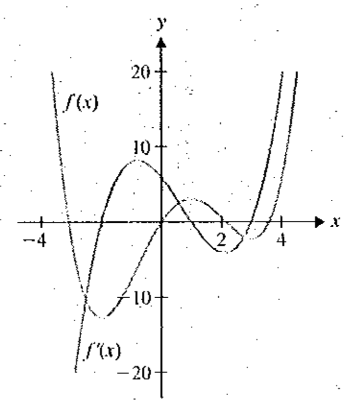


FIGURE 2.17
$y = f'(x)$ and a plausible graph of $y = f(x)$

superimposed on the graph of $y = f'(x)$. Further, $f'(x) < 0$ on the interval $(1, 3)$, so that the function decreases here. Finally, for $x > 3$, we have that $f'(x) > 0$, so that the function is increasing here. We show a graph exhibiting all of this behavior in Figure 2.17. We drew the graph of f so that the small "valley" on the right side of the y-axis was not as deep as the one on the left side of the y-axis for a reason. Look carefully at the graph of $f'(x)$ and notice that $|f'(x)|$ gets much larger on $(-2, 1)$ than on $(1, 3)$. This says that the tangent lines and hence, the graph will be much steeper on the interval $(-2, 1)$ than on $(1, 3)$. ■

HISTORICAL NOTES

Gottfried Leibniz (1646–1716) A German mathematician and philosopher who introduced much of the notation and terminology in calculus and who is credited (together with Sir Isaac Newton) with inventing the calculus. Leibniz was a prodigy who had already received his law degree and published papers on logic and jurisprudence by age 20. A true Renaissance man, Leibniz made important contributions to politics, philosophy, theology, engineering, linguistics, geology, architecture and physics, while earning a reputation as the greatest librarian of his time. Mathematically, he derived many fundamental rules for computing derivatives and helped promote the development of calculus through his extensive communications. The simple and logical notation he invented made calculus accessible to a wide audience and has only been marginally improved upon in the intervening 300 years. He wrote, "In symbols one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly . . . then indeed the labor of thought is wonderfully diminished."

○ Alternative Derivative Notations

We have denoted the derivative function by $f'(x)$. There are other commonly used notations, each with advantages and disadvantages. One of the coinventors of the calculus, Gottfried Leibniz, used the notation $\dfrac{df}{dx}$ (*Leibniz notation*) for the derivative. If we write $y = f(x)$, the following are all alternatives for denoting the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x).$$

The expression $\dfrac{d}{dx}$ is called a **differential operator** and tells you to take the derivative of whatever expression follows.

In section 2.1, we observed that $f(x) = |x|$, does not have a tangent line at $x = 0$ (i.e., it is not differentiable at $x = 0$), although it is continuous everywhere. Thus, there are continuous functions that are not differentiable. You might have already wondered whether the reverse is true. That is, are there differentiable functions that are not continuous? The answer (no) is provided by Theorem 2.1.

THEOREM 2.1

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

PROOF

For f to be continuous at $x = a$, we need only show that $\lim_{x \to a} f(x) = f(a)$. We consider

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a}(x - a) \right] \qquad \text{Multiply and divide by } (x - a).$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right] \lim_{x \to a}(x - a) \qquad \text{By Theorem 3.1 (iii) from section 1.3.}$$

$$= f'(a)(0) = 0, \qquad \text{Since } f \text{ is differentiable at } x = a.$$

where we have used the alternative definition of derivative (2.2) discussed earlier. By Theorem 3.1 in section 1.3, it now follows that

$$0 = \lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} f(x) - \lim_{x \to a} f(a)$$

$$= \lim_{x \to a} f(x) - f(a),$$

which gives us the result. ■

Note that Theorem 2.1 says that if a function is *not* continuous at a point then it *cannot* have a derivative at that point. It also turns out that functions are not differentiable at any point where their graph has a "sharp" corner, as is the case for $f(x) = |x|$ at $x = 0$. (See example 1.7.)

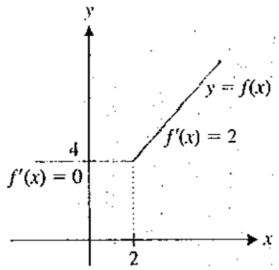


FIGURE 2.18
A sharp corner

EXAMPLE 2.7 Showing That a Function Is Not Differentiable at a Point

Show that $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ is not differentiable at $x = 2$.

Solution The graph (see Figure 2.18) indicates a sharp corner at $x = 2$, so you might expect that the derivative does not exist. To verify this, we investigate the derivative by evaluating one-sided limits. For $h > 0$, note that $(2 + h) > 2$ and so, $f(2 + h) = 2(2 + h)$. This gives us

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0^+} \frac{2(2+h) - 4}{h} \\ &= \lim_{h \to 0^+} \frac{4 + 2h - 4}{h} \\ &= \lim_{h \to 0^+} \frac{2h}{h} = 2. \end{aligned}$$

Likewise, if $h < 0$, $(2 + h) < 2$ and so, $f(2 + h) = 4$. Thus, we have

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{4 - 4}{h} = 0.$$

Since the one-sided limits do not agree ($0 \neq 2$), $f'(2)$ does not exist (i.e., f is not differentiable at $x = 2$). ■

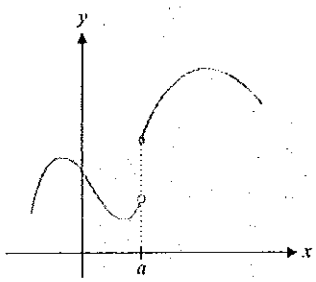


FIGURE 2.19a
A jump discontinuity

Figures 2.19a–2.19d show a variety of functions for which $f'(a)$ does not exist. In each case, convince yourself that the derivative does not exist.

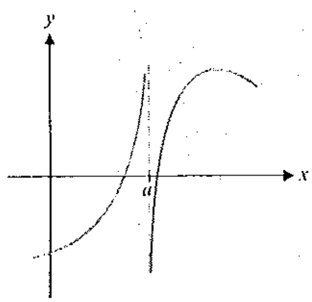


FIGURE 2.19b
A vertical asymptote

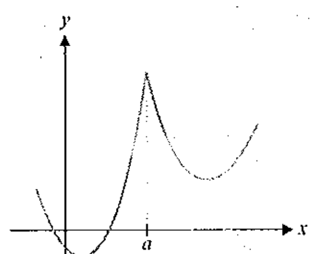


FIGURE 2.19c
A cusp

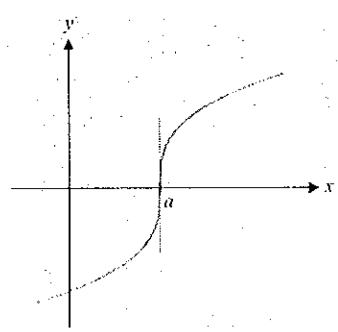


FIGURE 2.19d
A vertical tangent line

○ Numerical Differentiation

There are many times in applications when it is not possible or practical to compute derivatives symbolically. This is frequently the case in applications where we have only some data (i.e., a table of values) representing an otherwise *unknown* function. You will need an understanding of the limit definition to compute reasonable estimates of the derivative.

EXAMPLE 2.8 Approximating a Derivative Numerically

Numerically estimate the derivative of $f(x) = x^2\sqrt{x^3+2}$ at $x = 1$.

Solution We are not anxious to struggle through the limit definition for this function. The definition tells us, however, that the derivative at $x = 1$ is the limit of slopes of secant lines. We compute some of these below:

h	$\frac{f(1+h)-f(1)}{h}$
0.1	4.7632
0.01	4.3715
0.001	4.3342

h	$\frac{f(1+h)-f(1)}{h}$
−0.1	3.9396
−0.01	4.2892
−0.001	4.3260

Notice that the slopes seem to be converging to approximately 4.33 as h approaches 0. Thus, we make the approximation $f'(1) \approx 4.33$.

EXAMPLE 2.9 Estimating Velocity Numerically

Suppose that a sprinter reaches the following distances in the given times. Estimate the velocity of the sprinter at the 6-second mark.

$t(s)$	5.0	5.5	5.8	5.9	6.0	6.1	6.2	6.5	7.0
$f(t)$ (ft)	123.7	141.01	151.41	154.90	158.40	161.92	165.42	175.85	193.1

Solution The instantaneous velocity is the limit of the average velocity as the time interval shrinks. We first compute the average velocities over the shortest intervals given, from 5.9 to 6.0 and from 6.0 to 6.1.

Time Interval	*Average Velocity*
(5.9, 6.0)	35.0 ft/s
(6.0, 6.1)	35.2 ft/s

Time Interval	*Average Velocity*
(5.5, 6.0)	34.78 ft/s
(5.8, 6.0)	34.95 ft/s
(5.9, 6.0)	35.00 ft/s
(6.0, 6.1)	35.20 ft/s
(6.0, 6.2)	35.10 ft/s
(6.0, 6.5)	34.90 ft/s

Since these are the best individual estimates available from the data, we could just split the difference and estimate a velocity of 35.1 ft/s. However, there is useful information in the rest of the data. Based on the accompanying table, we can conjecture that the sprinter was reaching a peak speed at about the 6-second mark. Thus, we might accept the higher estimate of 35.2 ft/s. We should emphasize that there is not a single correct answer to this question, since the data are incomplete (i.e., we know the distance only at fixed times, rather than over a continuum of times).

BEYOND FORMULAS

In sections 2.3–2.8, we derive numerous formulas for computing derivatives. As you learn these formulas, keep in mind the reasons that we are interested in the derivative. Careful studies of the slope of the tangent line to a curve and the velocity of a moving object led us to the same limit, which we named the *derivative*. In general, the derivative represents the rate of change or the ratio of the change of one quantity to the change in another quantity. The study of change in a quantifiable way led directly to modern science and engineering. If we were limited to studying phenomena with only constant change, how much of the science that you have learned would still exist?

EXERCISES 2.2

WRITING EXERCISES

1. The derivative is important because of its many different uses and interpretations. Describe four aspects of the derivative: graphical (think of tangent lines), symbolic (the derivative function), numerical (approximations) and applications (velocity and others).
2. Mathematicians often use the word "smooth" to describe functions with certain (desirable) properties. Graphically, how are differentiable functions smoother than functions that are continuous but not differentiable, or functions that are not continuous?
3. Briefly describe what the derivative tells you about the original function. In particular, if the derivative is positive at a point, what do you know about the trend of the function at that point? What is different if the derivative is negative at the point?
4. Show that the derivative of $f(x) = 3x - 5$ is $f'(x) = 3$. Explain in terms of slope why this is true.

In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

1. $f(x) = 3x + 1, a = 1$
2. $f(x) = 3x^2 + 1, a = 1$
3. $f(x) = \sqrt{3x+1}, a = 1$
4. $f(x) = \dfrac{3}{x+1}, a = 2$

In exercises 5–12, compute the derivative function $f'(x)$ using (2.1) or (2.2).

5. $f(x) = 3x^2 + 1$
6. $f(x) = x^2 - 2x + 1$
7. $f(x) = \dfrac{3}{x+1}$
8. $f(x) = \dfrac{2}{2x-1}$
9. $f(x) = \sqrt{3x+1}$
10. $f(x) = 2x + 3$
11. $f(x) = x^3 + 2x - 1$
12. $f(x) = x^4 - 2x^2 + 1$

In exercises 13–18, match the graphs of the functions on the left with the graphs of their derivatives on the right.

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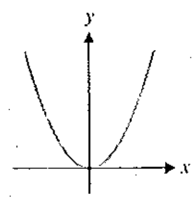

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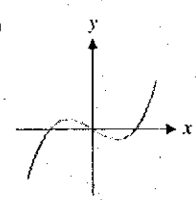

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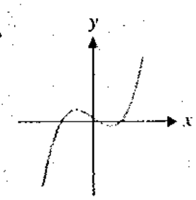

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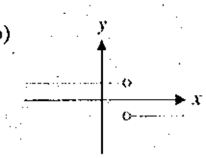

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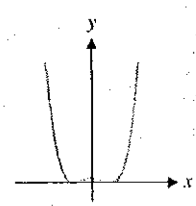

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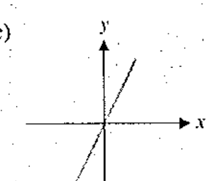

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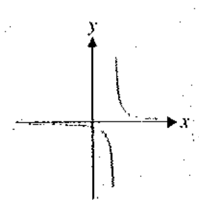

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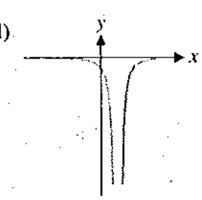

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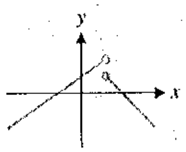

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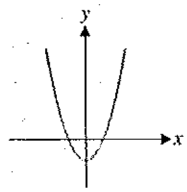
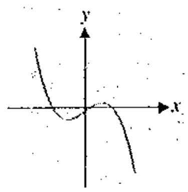
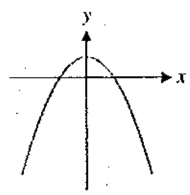


18.

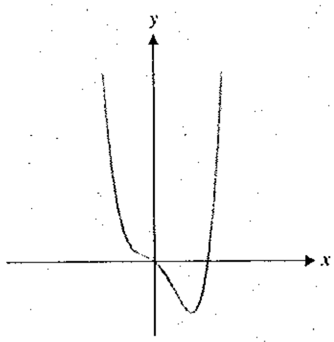


(f)



In exercises 19–22, use the given graph of $f(x)$ to sketch a graph of $f'(x)$.

19.



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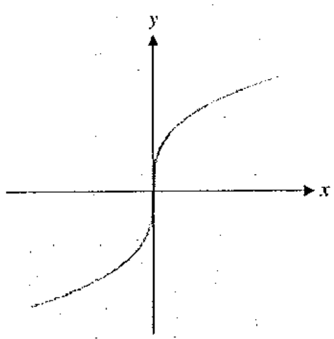

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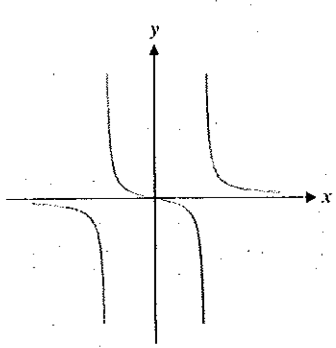

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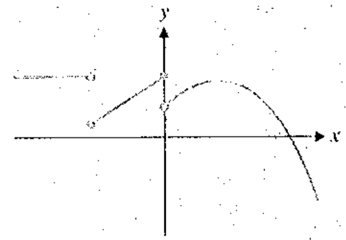
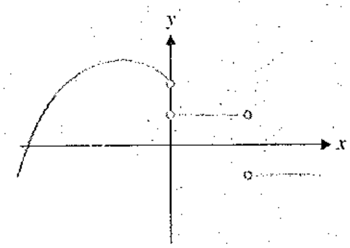


In exercises 23 and 24, use the given graph of $f'(x)$ to sketch a plausible graph of a continuous function $f(x)$.

23.



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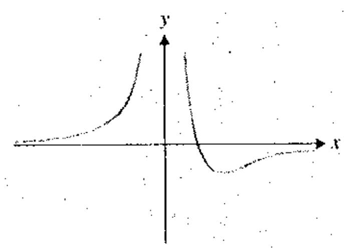


25. Graph $f(x) = |x| + |x - 2|$ and identify all x-values at which $f(x)$ is not differentiable.

26. Graph $f(x) = e^{-2/(x^3-x)}$ and identify all x-values at which $f(x)$ is not differentiable.

27. Find all real numbers p such that $f'(0)$ exists for $f(x) = x^p$.

28. Prove that if $f(x)$ is differentiable at $x = a$, then $\lim_{h \to 0} \frac{f(a + ch) - f(a)}{h} = cf'(a)$.

29. If $f(x)$ is differentiable at $x = a \neq 0$, evaluate $\lim_{x \to a} \frac{f(x^2) - f(a^2)}{x^2 - a^2}$.

30. Prove that if $f(x)$ is differentiable at $x = 0$, $f(x) \leq 0$ for all x and $f(0) = 0$, then $f'(0) = 0$.

31. The table shows the margin of error in degrees for tennis serves hit at 100 mph with various amounts of topspin (in units of revolutions per second). Estimate the slope of the derivative at $x = 60$, and interpret it in terms of the benefit of extra spin. (Data adapted from *The Physics and Technology of Tennis* by Brody, Cross and Lindsey.)

Topspin (rps)	20	40	60	80	100
Margin of error	1.8	2.4	3.1	3.9	4.6

32. The table shows the margin of error in degrees for tennis serves hit at 120 mph from various heights. Estimate the slope of the derivative at $x = 8.5$ and interpret it in terms of hitting a serve from a higher point. (Data adapted from *The Physics and Technology of Tennis* by Brody, Cross and Lindsey.)

Height (ft)	7.5	8.0	8.5	9.0	9.5
Margin of error	0.3	0.58	0.80	1.04	1.32

In exercises 33 and 34, use the distances $f(t)$ to estimate the velocity at $t = 2$.

33.

t	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(t)$	3.1	3.9	4.8	5.8	6.8	7.7	8.5

34.

t	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(t)$	4.6	5.3	6.1	7.0	7.8	8.6	9.3

35. The Environmental Protection Agency uses the measurement of ton-MPG to evaluate the power-train efficiency of vehicles. The ton-MPG rating of a vehicle is given by the weight of the vehicle (in tons) multiplied by a rating of the vehicle's fuel efficiency in miles per gallon. Several years of data for new cars are given in the table. Estimate the rate of change of ton-MPG in (a) 1994 and (b) 2000. Do your estimates imply that cars are becoming more or less efficient? Is the rate of change constant or changing?

Year	1992	1994	1996	1998	2000
Ton-MPG	44.9	45.7	46.5	47.3	47.7

36. The fuel efficiencies in miles per gallon of cars from 1992 to 2000 are shown in the following table. Estimate the rate of change in MPG in (a) 1994 and (b) 2000. Do your estimates imply that cars are becoming more or less fuel efficient? Comparing your answers to exercise 35, what must be happening to the average weight of cars? If weight had remained constant, what do you expect would have happened to MPG?

Year	1992	1994	1996	1998	2000
MPG	28.0	28.1	28.3	28.5	28.1

In exercises 37 and 38, use a CAS or graphing calculator.

37. Numerically estimate $f'(1)$ for $f(x) = x^x$ and verify your answer using a CAS.

38. Numerically estimate $f'(\pi)$ for $f(x) = x^{\sin x}$ and verify your answer using a CAS.

In exercises 39 and 40, compute the right-hand derivative $D_+f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ and the left-hand derivative $D_-f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$.

39. $f(x) = \begin{cases} 2x+1 & \text{if } x < 0 \\ 3x+1 & \text{if } x \geq 0 \end{cases}$

40. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$

41. Assume that $f(x) = \begin{cases} g(x) & \text{if } x < 0 \\ k(x) & \text{if } x \geq 0 \end{cases}$. If f is continuous at $x = 0$ and g and k are differentiable at $x = 0$, prove that $D_+f(0) = k'(0)$ and $D_-f(0) = g'(0)$. Which statement is not true if f has a jump discontinuity at $x = 0$?

42. Explain why the derivative $f'(0)$ exists if and only if the one-sided derivatives exist and are equal.

43. If $f'(x) > 0$ for all x, use the tangent line interpretation to argue that f is an **increasing function;** that is, if $a < b$, then $f(a) < f(b)$.

44. If $f'(x) < 0$ for all x, use the tangent line interpretation to argue that f is a **decreasing function;** that is, if $a < b$, then $f(a) > f(b)$.

45. If $f(x) = x^{2/3}$, show graphically and numerically that f is continuous at $x = 0$ but $f'(0)$ does not exist.

46. If $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$, show graphically and numerically that f is continuous at $x = 0$ but $f'(0)$ does not exist.

47. Give an example showing that the following is not true for all functions f: if $f(x) \leq x$, then $f'(x) \leq 1$.

48. Determine whether the following is true for all functions f: if $f(0) = 0$, $f'(x)$ exists for all x and $f(x) \leq x$, then $f'(x) \leq 1$.

In exercises 49 and 50, give the units for the derivative function.

49. (a) $f(t)$ represents position, measured in meters, at time t seconds.
(b) $f(x)$ represents the demand, in number of items, of a product when the price is x dollars.

50. (a) $c(t)$ represents the amount of a chemical present, in grams, at time t minutes.
(b) $p(x)$ represents the mass, in kg, of the first x meters of a pipe.

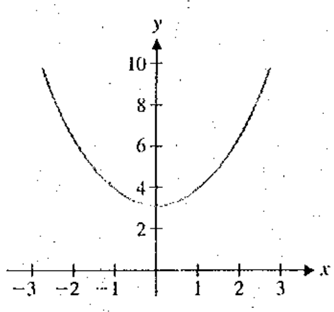
51. Let $f(t)$ represent the trading value of a stock at time t days. If $f'(t) < 0$, what does that mean about the stock? If you held some shares of this stock, should you sell what you have or buy more?

52. Suppose that there are two stocks with trading values $f(t)$ and $g(t)$, where $f(t) > g(t)$ and $0 < f'(t) < g'(t)$. Based on this information, which stock should you buy? Briefly explain.

53. One model for the spread of a disease assumes that at first the disease spreads very slowly, gradually the infection rate increases to a maximum and then the infection rate decreases back to zero, marking the end of the epidemic. If $I(t)$ represents the number of people infected at time t, sketch a graph of both $I(t)$ and $I'(t)$, assuming that those who get infected do not recover.

54. One model for urban population growth assumes that at first, the population is growing very rapidly, then the growth rate decreases until the population starts decreasing. If $P(t)$ is the population at time t, sketch a graph of both $P(t)$ and $P'(t)$.

55. Use the graph to list the following in increasing order: $f(1)$, $f(2) - f(1)$, $\dfrac{f(1.5) - f(1)}{0.5}$, $f'(1)$.



Exercises 55 and 56

56. Use the graph to list the following in increasing order: $f(0)$, $f(0) - f(-1)$, $\dfrac{f(0) - f(-0.5)}{0.5}$, $f'(0)$.

In exercises 57–60, the limit equals $f'(a)$ for some function $f(x)$ and some constant a. Determine $f(x)$ and a.

57. $\displaystyle\lim_{h\to 0} \frac{(1+h)^2 - (1+h)}{h}$

58. $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$

59. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$

60. $\displaystyle\lim_{h\to 0} \frac{(h-1)^2 - 1}{h}$

61. Sketch the graph of a function with the following properties: $f(0) = 1$, $f(1) = 0$, $f(3) = 6$, $f'(0) = 0$, $f'(1) = -1$ and $f'(3) = 4$.

62. Sketch the graph of a function with the following properties: $f(-2) = 4$, $f(0) = -2$, $f(2) = 1$, $f'(-2) = -2$, $f'(0) = 2$ and $f'(2) = 1$.

63. A phone company charges one dollar for the first 20 minutes of a call, then 10 cents per minute for the next 60 minutes and 8 cents per minute for each additional minute (or partial minute). Let $f(t)$ be the price in cents of a t-minute phone call, $t > 0$. Determine $f'(t)$ as completely as possible.

64. The table shows the percentage of English Premier League soccer players by birth month, where $x = 0$ represents November, $x = 1$ represents December and so on. (The data are adapted from John Wesson's *The Science of Soccer*.) If these data come from a differentiable function $f(x)$, estimate $f'(1)$. Interpret the derivative in terms of the effect of being a month older but in the same grade of school.

Month	0	1	2	3	4
Percent	13	11	9	7	7

EXPLORATORY EXERCISES

1. Compute the derivative function for x^2, x^3 and x^4. Based on your results, identify the pattern and conjecture a general formula for the derivative of x^n. Test your conjecture on the functions $\sqrt{x} = x^{1/2}$ and $1/x = x^{-1}$.

2. In Theorem 2.1, it is stated that a differentiable function is guaranteed to be continuous. The converse is not true: continuous functions are not necessarily differentiable (see example 2.7). This fact is carried to an extreme in **Weierstrass' function**, to be explored here. First, graph the function $f_4(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x + \frac{1}{16}\cos 81x$ in the graphing window $0 \le x \le 2\pi$ and $-2 \le y \le 2$. Note that the graph appears to have several sharp corners, where a derivative would not exist. Next, graph the function $f_6(x) = f_4(x) + \frac{1}{32}\cos 243x + \frac{1}{64}\cos 729x$. Note that there are even more places where the graph appears to have sharp corners. Explore graphs of $f_{10}(x)$, $f_{14}(x)$ and so on, with more terms added. Try to give graphical support to the fact that the Weierstrass function $f_\infty(x)$ is continuous for all x but is not differentiable for any x. More graphical evidence comes from the fractal nature of the Weierstrass function: compare the graphs of $f_4(x)$ with $0 \le x \le 2\pi$ and $-2 \le y \le 2$ and $f_6(x) - \cos x - \frac{1}{2}\cos 3x$ with $0 \le x \le \frac{2\pi}{9}$ and $-\frac{1}{2} \le y \le \frac{1}{2}$. Explain why the graphs are identical. Then explain why this indicates that no matter how much you zoom in on a graph of the Weierstrass function, you will continue to see wiggles and corners. That is, you cannot zoom in to find a tangent line.

3. Suppose there is a function $F(x)$ such that $F(1) = 1$ and $F(0) = f_0$, where $0 < f_0 < 1$. If $F'(1) > 1$, show graphically that the equation $F(x) = x$ has a solution q where $0 < q < 1$. (Hint: Graph $y = x$ and a plausible $F(x)$ and look for

intersections.) Sketch a graph where $F'(1) < 1$ and there are no solutions to the equation $F(x) = x$ between 0 and 1 (although $x = 1$ is a solution). Solutions have a connection with the probability of the extinction of animals or family names. Suppose you and your descendants have children according to the following probabilities: $f_0 = 0.2$ is the probability of having no children, $f_1 = 0.3$ is the probability of having exactly one child, and $f_2 = 0.5$ is the probability of having two children. Define $F(x) = 0.2 + 0.3x + 0.5x^2$ and show that $F'(1) > 1$. Find the solution of $F(x) = x$ between $x = 0$ and $x = 1$; this number is the probability that your "line" will go extinct some time into the future. Find nonzero values of f_0, f_1 and f_2 such that the corresponding $F(x)$ satisfies $F'(1) < 1$ and hence the probability of your line going extinct is 1.

4. The **symmetric difference quotient** of a function f centered at $x = a$ has the form $\frac{f(a+h) - f(a-h)}{2h}$. If $f(x) = x^2 + 1$ and $a = 1$, illustrate the symmetric difference quotient as a slope of a secant line for $h = 1$ and $h = 0.5$. Based on your picture, conjecture the limit of the symmetric difference quotient as h approaches 0. Then compute the limit and compare to the derivative $f'(1)$ found in example 1.1. For $h = 1$, $h = 0.5$ and $h = 0.1$, compare the actual values of the symmetric difference quotient and the usual difference quotient $\frac{f(a+h) - f(a)}{h}$. In general, which difference quotient provides a better estimate of the derivative? Next, compare the values of the difference quotients with $h = 0.5$ and $h = -0.5$ to the derivative $f'(1)$. Explain graphically why one is smaller and one is larger. Compare the average of these two difference quotients to the symmetric difference quotient with $h = 0.5$. Use this result to explain why the symmetric difference quotient might provide a better estimate of the derivative. Next, compute several symmetric difference quotients of $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ centered at $a = 2$. Recall that in example 2.7 we showed that the derivative $f'(2)$ does not exist. Given this, discuss one major problem with using the symmetric difference quotient to approximate derivatives. Finally, show that if $f'(a)$ exists, then $\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$.

2.3 COMPUTATION OF DERIVATIVES: THE POWER RULE

You have now computed numerous derivatives using the limit definition. In fact, you may have computed enough that you have started taking some shortcuts. In exploratory exercise 1 in section 2.2, we asked you to compile a list of derivatives of basic functions and to generalize. We continue that process in this section.

The Power Rule

We first revisit the limit definition of derivative to compute two very simple derivatives.

For any constant c,
$$\frac{d}{dx} c = 0. \tag{3.1}$$

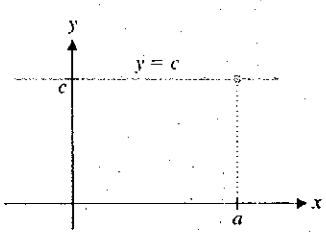


FIGURE 2.20
A horizontal line

Notice that (3.1) says that for any constant c, the horizontal line $y = c$ has a tangent line with zero slope. That is, the tangent line to a horizontal line is the same horizontal line (see Figure 2.20).

Let $f(x) = c$, for all x. From the definition in equation (2.3), we have

$$\frac{d}{dx}c = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$